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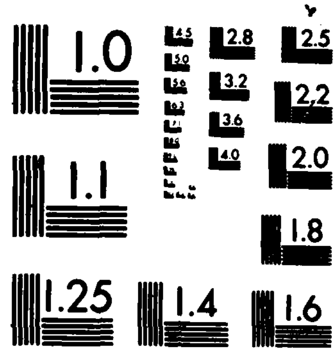

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STABILITY LIMITS OF MODIFIED HILBER-HUGHES ALGORITHM FOR TIME INTEGRATION IN DYNAMIC ANALYSIS

BY MINOS MOUSSOUROS

RESEARCH AND TECHNOLOGY DEPARTMENT

AUGUST 1986

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This report extends the range of unconditional stability of a modified form of the implicit method of Hilber-Hughes. The stability analysis is applicable only to the uncoupled linear modal equations. Tables and curves are presented for a convenient choice of the involved parameters. [illegible]

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FOREWORD

This report presents an extension of a modified form of the implicit method of Hilber-Hughes, pertaining to the temporal integration of the equilibrium equations of motion of a structural system. The unconditional stability characteristics of the uncoupled linear equations are given and their range extended. Tables and curves are presented for "a convenient choice" of the involved parameters.

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CHAPTER 1

INTRODUCTION

The analysis of time dependent problems in fluid and structural mechanics through finite element or difference methods involves temporal discretization of the basic conservation laws. In structural mechanics, this leads to time integration techniques of Newton's second law of motion. A survey of previous work reveals an abundance of references.[1-44] This report is limited to structural engineering work.[1-40] Reference 41 is an example of fluid dynamics time integration schemes. Discretization methods are typically divided in two different ways. In the first, we have what is referred to in Reference 45 as linear singlestep or multistep methods. In singlestep methods only variables at the current and next time steps are considered. In multistep methods unknowns at more than one step are involved. In the second, time discretization schemes are classified in two large groups depending on stability and need of inversion characteristics. These are implicit methods and explicit methods.

Implicit methods require inversion of large matrices and can be unconditionally or conditionally stable. They are predominantly used in structural dynamics, where low-mode response is of interest. Naturally, preference is given to unconditionally stable methods due to their unrestrictiveness. If conditionally stable, the time step cannot exceed a limiting time step or the method becomes unattractive.

Explicit methods such as the central difference method (References 1, 3, and 8) are preferred in solving wave propagation problems. These methods depend on what is referred to as the "critical time step" or "Courant number." Use of larger time steps cause the solution to become unstable. In general, the spatial mesh is not uniform; therefore, the time step will depend on the smaller element. The "smaller" element dimensions put a ceiling on the "highest admissible" frequency of structural response and a "cutoff frequency" is introduced. In addition, nonuniform meshes introduce spurious frequencies of reflection (References 42-44) from the various element interfaces.

The question of mesh size introduces the "cutoff" frequency concept (already mentioned for explicit methods); it also applies to implicit methods.

Examples of structural dynamics time integrators, among others [Houbolt's method (References 1 and 30) and Park's method (Reference 3)] are:

(1) the Newmark family of methods (References 1 and 9);

(2) the Wilson θ-method (References 1, 11, and 12); and

(3) the Hilber-Hughes collocation method (References 15, 16, and 17) (to distinguish from their α-method).

Table 1-1 summarizes the Newmark, Wilson θ, Hilber-Hughes collocation, and α-methods, respectively, together with their possible extension (the modified Hilber-Hughes method).

In constructing a discrete operator capable of modeling the continuous system, the following point should be kept in mind. The algorithm should be unconditionally stable, and dissipative of spurious higher modes but not lower modes. (This damping is of numerical nature, introduced by temporal discretization, and unrelated to the actual structural damping.)

The above properties can only be verified for linear problems. Very little is known for nonlinear problems.

Table 1-2 summarizes the range for unconditional stability of these methods with additional notes. Note that when $\beta = 0$ and $\gamma = .50$, the Newmark method becomes the "central difference" explicit method; when $\beta = 1.0$ and $\gamma = 1.5$, the scheme is a backward difference; and when $\beta = .25$ and $\gamma = .50$, we obtain the "average acceleration" method. Other combinations can be found in Reference 2.

All three methods use some approximation for acceleration, velocity, and displacement vectors of the form

$$\ddot{x}_{n+\theta} = G_1(\ddot{x}_n, \theta) \tag{1-1}$$

$$\dot{x}_{n+\theta} = G_2(\dot{x}_n, \ddot{x}_n, \ddot{x}_{n+1}, \theta, \gamma, \Delta t) \tag{1-2}$$

$$x_{n+\theta} = G_3(x_n, \dot{x}_n, \ddot{x}_n, \ddot{x}_{n+1}, \theta, \gamma, \Delta t) \tag{1-3}$$

where subscript n indicates point in time, t_n; $n+\theta$ is point in time, $t_{n+\theta}$ (θ can take the value of 1); Δt is the time increment; and dots denote differentiation with respect to time. Zienkiewicz and Wood (References 2, 28, 29, and 31) employ a weighted residual formulation to come up with the various temporal approximation schemes.

The analysis is concluded by satisfying the continuum equations of equilibrium

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{x}} + \underset{\sim}{C}\,\dot{\underset{\sim}{x}} + \underset{\sim}{K}\,\underset{\sim}{x} = \underset{\sim}{F}(t) \tag{1-4}$$

by their discrete analogue of some point in time $n+\theta$,

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{x}}_{n+\theta} + \underset{\sim}{C}\,\dot{\underset{\sim}{x}}_{n+\theta} + (1+\alpha)\underset{\sim}{K}\,x_{n+\theta} - \alpha\underset{\sim}{K}\,\underset{\sim}{x}_{n+\theta} = \underset{\sim}{F}_{n+\theta}, \tag{1-5}$$

where α is a new parameter that may be 0, 1, or negative. Here $\underset{\sim}{M}$ is the assembled mass matrix (typically positive definite and symmetric), $\underset{\sim}{C}$ the structural damping matrix (usually of Rayleigh form), $\underset{\sim}{K}$ the stiffness matrix (positive semidefinite), and $\underset{\sim}{F}(t)$ the external loading vector.

These methods have introduced parameters α, β, γ, and θ apart from time step Δt to control stability, dissipation, and period elongation (i.e. control "distortion" effects).

Typically, the study of the stability characteristics of these methods can be summarized in the following logical steps:

1. The system [Equation (1-5)] is assumed linear and can be decoupled by a normal mode procedure.[1] The resulting equation is of the form

$$m\,\ddot{x}_{n+\theta} + c\,\dot{x}_{n+\theta} + (1+\alpha)k\,x_{n+\theta} - \alpha k\,x_{n+\theta} = f_{n+\theta} \tag{1-6}$$

In the analysis of the Newmark β-method and Wilson θ-methods, the structural damping c can be left in because the expressions do not become extremely complicated. The same is not true for the α-Hughes method. The analysis of the stability characteristics of the linear system is for the undamped system ($c = 0$).

2. Use of Equations (1-1), (1-2), (1-3), and (1-6) results in a (3 x 3) matrix equation relating the historical variables $\underset{\sim}{x}_n$ with the variables at the next time point, i.e.

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}\,\underset{\sim}{X}_n + \underset{\sim}{L}\,F_{n+\theta}$$

where the state column vectors $\underset{\sim}{X}_{n+1}$ and $\underset{\sim}{X}_n$ are defined by

$$\underset{\sim}{X}_{n+1} = \begin{bmatrix} \Delta t^2\ \ddot{x}_{n+1} \\ \Delta t\ \dot{x}_{n+1} \\ x_{n+1} \end{bmatrix}$$

$$\underset{\sim}{X}_{n} = \begin{bmatrix} \Delta t^2\ \ddot{x}_{n} \\ \Delta t\ \dot{x}_{n} \\ x_{n} \end{bmatrix}$$

and $\underset{\sim}{A}$ is an (3 x 3) operator matrix, the so-called amplification matrix; $\underset{\sim}{L}$ is a (3 x 1) column vector; and $F_{n+\theta}$ is a scalar.

3. Analysis of the eigenvalue properties of $\underset{\sim}{A}$ using the Routh-Hurwitz criterion (Reference 45) and the Cayley-Hamilton theorem (Reference 46) allows us to define the region of "unconditional stability" of the linear system. Naturally, no eigenvalue can exceed 1 or the solution becomes unbounded. This is the process for establishing good properties for the temporal operator $\underset{\sim}{A}$.

TABLE 1-1. SUMMARY OF EMPLOYED EQUATIONS FOR NEWMARK β, WILSON Θ, AND HILBER-HUGHES METHODS

METHOD	EQUATION TYPE	EQUATION
NEWMARK β	EQUILIBRIUM	$\ddot{\chi}_{n+1} + 2\xi\omega\dot{\chi}_{n+1} + \omega^2\chi_{n+1} = f_{n+1}$
	DISPLACEMENT	$\chi_{n+1} = \chi_n + \Delta t\dot{\chi}_n + (\Delta t)^2\left[\left(\frac{1}{2} - \beta\right)\ddot{\chi}_n + \beta\ddot{\chi}_{n+1}\right]$
	VELOCITY	$\dot{\chi}_{n+1} = \dot{\chi}_n + \Delta t\left[(1-\gamma)\ddot{\chi}_n + \gamma\ddot{\chi}_{n+1}\right]$
	ACCELERATION	
WILSON Θ	EQUILIBRIUM	$\ddot{\chi}_{n+\Theta} + 2\xi\omega\dot{\chi}_{n+\Theta} + \omega^2\chi_{n+\Theta} = f_{n+\Theta}$
	DISPLACEMENT	$\chi_{n+\Theta} = \chi_n + (\Theta\Delta t)\dot{\chi}_n + (\Theta\Delta t)^2\left[\left(\frac{1}{2} - \frac{1}{6}\Theta\right)\ddot{\chi}_n + \frac{1}{6}\Theta\ddot{\chi}_{n+1}\right]$
	VELOCITY	$\dot{\chi}_{n+\Theta} = \dot{\chi}_n + (\Theta\Delta t)\left[\left(1 - \frac{\Theta}{2}\right)\ddot{\chi}_n + \frac{1}{2}\Theta\ddot{\chi}_{n+1}\right]$
	ACCELERATION	$\ddot{\chi}_{n+\Theta} = (1-\Theta)\ddot{\chi}_n + \Theta\ddot{\chi}_{n+1}$
HILBER-HUGHES COLLOCATION METHOD	EQUILIBRIUM	$m\ddot{\chi}_{n+\Theta} + c\dot{\chi}_{n+\Theta} + k\chi_{n+\Theta} = f_{n+\Theta}$
	DISPLACEMENT	$\chi_{n+\Theta} = \chi_n + (\Theta\Delta t)\dot{\chi}_n + (\Theta\Delta t)^2\left[\left(\frac{1}{2} - \beta\right)\ddot{\chi}_n + \beta\ddot{\chi}_{n+\Theta}\right]$ $\chi_{n+1} = \chi_n + \Delta t\dot{\chi}_n + \Delta t^2\left[\left(\frac{1}{2} - \beta\right)\ddot{\chi}_n + \beta\ddot{\chi}_{n+1}\right]$
	VELOCITY	$\dot{\chi}_{n+1} = \dot{\chi}_n + \Delta t\left[(1-\gamma)\ddot{\chi}_n + \gamma\ddot{\chi}_{n+1}\right]$
	ACCELERATION	$\ddot{\chi}_{n+\Theta} = (1-\Theta)\ddot{\chi}_n + \Theta\ddot{\chi}_{n+1}$

TABLE 1-1. (CONT.)

METHOD	EQUATION TYPE	EQUATION
HILBER-HUGHES α - METHOD	EQUILIBRIUM	$m\ddot{\chi}_{n+1} + c\dot{\chi}_{n+1} + (1+\alpha)k\chi_{n+1} - \alpha k\chi_n = f_{n+1}$
	DISPLACEMENT	$\chi_{n+1} = \chi_n + \Delta t\dot{\chi}_n + \Delta t^2\left[\left(\frac{1}{2}-\beta\right)\ddot{\chi}_n + \beta\ddot{\chi}_{n+1}\right]$
	VELOCITY	$\dot{\chi}_{n+1} = \dot{\chi}_n + \Delta t\left[(1-\gamma)\ddot{\chi}_n + \gamma\ddot{\chi}_{n+1}\right]$
	ACCELERATION	
MODIFIED HILBER-HUGHES METHOD	EQUILIBRIUM	$m\ddot{\chi}_{n+\Theta} + c\dot{\chi}_{n+\Theta} + (1+\alpha)k\chi_{n+\Theta} - \alpha k\chi_n = f_{n+\Theta}$ or $\ddot{\chi}_{n+\Theta} + 2\omega\xi\dot{\chi}_{n+\Theta} + (1+\alpha)\omega^2\dot{\chi}_{n+\Theta} - \alpha\omega^2\chi_n = \frac{1}{m}f_{n+\Theta}$
	DISPLACEMENT	$\chi_{n+\Theta} = \chi_n + (\Theta\Delta t)\dot{\chi}_n + (\Theta\Delta t)^2\left[\left(\frac{1}{2}-\beta\Theta\right)\ddot{\chi}_n + \beta\Theta\ddot{\chi}_{n+1}\right]$
	VELOCITY	$\dot{\chi}_{n+\Theta} = \dot{\chi}_n + (\Theta\Delta t)\left[(1-\gamma\Theta)\ddot{\chi}_n + \gamma\Theta\ddot{\chi}_{n+1}\right]$
	ACCELERATION	$\ddot{\chi}_{n+\Theta} = (1-\Theta)\ddot{\chi}_n + \Theta\ddot{\chi}_{n-1}$

TABLE 1-2. SUMMARY FOR UNCONDITIONAL STABILITY OF NEWMARK β, WILSON Θ, AND HILBER-HUGHES METHODS

METHOD	ABBREVIATIONS AND NOTES	CONDITIONS
NEWMARK β	(1) $\gamma = \frac{1}{2},\ \beta \geqslant \frac{1}{4},\ \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1$ ($x = \xi = 0$, NO STRUCTURAL DAMPING) (2) $\gamma = \frac{1}{2},\ \beta = \frac{1}{4},\ (x = \xi = 0),\ \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1,$ i.e. NO AMPLIFICATION OR DISSIPATION (3) $\gamma = \frac{3}{2},\ \beta = 1,\ (x = \xi = 0),\ \lambda_1 = 0,\ \lambda_2\lambda_3 = \frac{1}{1+s^2},\ s = \omega\Delta t = 2\pi\frac{\Delta t}{T}$ WITH DISSIPATION AT HIGHER MODES	$\beta \geqslant \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2$ $\gamma \geqslant \frac{1}{2}$ $(\frac{1}{2} - \gamma + \beta \geqslant 0$ (SUPERFLUOUS DUE TO FIRST TWO)
WILSON Θ - METHOD	$p = \frac{\Delta t}{T},\quad x = \frac{\xi B}{2\pi p},\quad B = \frac{24\pi^2 p^2}{[6\Theta + 12\pi\xi\Theta^2 p + 4\pi^2\Theta^3 p^2]}$ $\omega = \frac{2\pi}{T}$ $2\xi\omega = \frac{c}{m}$ WITH CHARACTERISTIC EQUATION IN λ $\lambda^3 - \left[3 - \frac{1}{\Theta} - (1+\Theta)x - \frac{1}{6}B - \frac{1}{2}B\Theta - \frac{1}{3}B\Theta^2\right]\lambda^2 -$ $-\left[-3 - \frac{2}{3}B + \frac{2}{3}B\Theta^2 + \frac{2}{\Theta} + 2x\Theta\right]\lambda -$ $-\left[1 + x + \frac{1}{2}B\Theta - \frac{1}{6}B - \frac{1}{3}B\Theta^2 - \frac{1}{\Theta} - x\Theta\right] = 0$	ROOTS $\lvert\lambda_i\rvert \leqslant 1$ ($i = 1, 2, 3$) $\Theta \geqslant 1.420815$
HILBER-HUGHES COLLOCATION METHOD	NONE	$\gamma = \frac{1}{2}$ $\Theta \geqslant 1$ $\frac{\Theta}{2(\Theta+1)} \geqslant \beta \geqslant \frac{2\Theta^2 - 1}{4(2\Theta^3 - 1)}$
HILBER-HUGHES α - METHOD	NONE	$\gamma = \frac{1}{2} - \alpha$ $\beta = \frac{1}{4}(1-\alpha)^2$ $-\frac{1}{3} \leqslant \alpha \leqslant 0$

CHAPTER 2

STABILITY OF HILBER-HUGHES COLLOCATION METHOD

Hilber and Hughes (References 15, 16, and 17) presented a new scheme for the integration of the linear dynamic equations of motion of structural problems. It is based on collocation, and its stability and analysis was developed for any undamped linear system.

As before, the initial value problem we seek to solve numerically can be expressed in the form

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{x}} + \underset{\sim}{K}\,\underset{\sim}{x} = \underset{\sim}{F}(t) \tag{2-1}$$

subject to the initial conditions at time t = 0

$$\underset{\sim}{x}(0) = \underset{\sim}{d} \tag{2-2}$$

$$\dot{\underset{\sim}{x}}(0) = \underset{\sim}{v}$$

and

$$\ddot{\underset{\sim}{x}}(0) = \underset{\sim}{M}^{-1}\,[\underset{\sim}{F}(0) - \underset{\sim}{K}\,\underset{\sim}{d}] \tag{2-3}$$

The object of this method is to select additional parameters apart from step size in such a way as to obtain unconditional stability and dissipation at high frequencies to control the response which, in the case of structural problems, depends on the low mode spectrum.

They proposed to satisfy the uncoupled version of the equations of motion [Equation (2-1)] at a point $n + \theta$, or $(n+\theta)\Delta t$, i.e.

$$m\,\ddot{x}_{n+\theta} + k\,x_{n+\theta} = f_{n+\theta} \tag{2-4}$$

The following approximations for accelerations, displacements, and external loading were used:

$$\ddot{x}_{n+\theta} = (1-\theta)\ddot{x}_n + \theta\,\ddot{x}_{n+1} \tag{2-5}$$

$$x_{n+\theta} = x_n + \theta\Delta t\,\dot{x}_n + (\theta\Delta t)^2\left[(\tfrac{1}{2}-\beta)\ddot{x}_n + \beta\,\ddot{x}_{n+\theta}\right] \tag{2-6}$$

$$f_{n+\theta} = (1-\theta)f_n + \theta\ f_{n+1} \tag{2-7}$$

$$x_{n+1} = x_n + \Delta t\ \dot{x}_n + \Delta t^2\left[(\tfrac{1}{2}-\beta)\ddot{x}_n + \beta\ \ddot{x}_{n+1}\right] \tag{2-8}$$

Velocities are obtained through a separate equation

$$\dot{x}_{n+1} = \dot{x}_n + \Delta t\left[(1-\gamma)\ \ddot{x}_n + \gamma\ \ddot{x}_{n+1}\right] \tag{2-9}$$

Substituting the expression for acceleration $\ddot{x}_{n+\theta}$ [Equation (2-5)] the expression for displacement $x_{n+\theta}$ [Equation (2-6)], and leaving the loading $f_{n+\theta}$ as it is in Equation (2-4), we solve for the acceleration $\ddot{x}_{n+1}$. We obtain Equation (2-10)

$$\theta m\left[1 + m^{-1}k\beta\theta^2\Delta t^2\right]\ddot{x}_{n+1} = f_{n+\theta} - k\ x_n - \theta\Delta t k\dot{x}_n - \\ m\left[(1-\theta) + \left\{(\tfrac{1}{2}-\beta)\theta^2 + \beta(1-\theta)\theta^2\right\}\ \Delta t^2 m^{-1}k\right]\ddot{x}_n \tag{2-10}$$

Abbreviating by

$$\omega^2 = k/m \tag{2-11}$$

$$\Omega = \omega\Delta t \tag{2-12}$$

$$D = \theta\left[1+\beta\theta^2\Omega^2\right] = \theta\left[1+m^{-1}k\beta\theta^2\Delta t^2\right] \tag{2-13}$$

and noting that

$$1 - \theta + (\tfrac{1}{2}-\beta)\theta^2\Omega^2 + \beta\theta^2\Omega^2(1-\theta) \\ = -\left[D - (1+\tfrac{1}{2}\theta^2\Omega^2)\right] \tag{2-14}$$

the expression for the acceleration at point n+1 in time can be expressed in terms of the acceleration at the previous step and other historical variables:

$$\Delta t^2\ddot{x}_{n+1} = \Delta t^2\left(\frac{f_{n+\theta}}{mD}\right) - \frac{\Omega^2}{D}x_n - \frac{\theta\Omega^2}{D}\Delta t\dot{x}_n + \\ \left\{1 - \frac{(1+\tfrac{1}{2}\theta^2\Omega^2)}{D}\right\}\Delta t^2\ddot{x}_n \tag{2-15}$$

Furthermore, by using Equation (2-9) to obtain the velocity at the next step in terms of the already known acceleration $\ddot{x}_{n+1}$ [Equation (2-15)], we get

$$\Delta t\dot{x}_{n+1} = \frac{\gamma\Omega^2}{\omega^2 mD} f_{n+\theta} - \frac{\gamma\Omega^2}{D} x_n + \left\{1 - \frac{\gamma\theta\Omega^2}{D}\right\}\Delta t\dot{x}_n + \Delta t^2\ddot{x}_n \left\{1 - \gamma\frac{(1+\frac{1}{2}\theta^2\Omega^2)}{D}\right\} \qquad (2\text{-}16)$$

Similarly, employing Equations (2-8) and (2-15) to obtain the displacements

$$x_{n+1} = \beta A_{30} f_{n+\theta} + (1+\beta A_{31})x_n + (1+\beta A_{32})\Delta t\dot{x}_n + \left\{\tfrac{1}{2} + \beta(A_{33}-1)\right\}\Delta t^2\ddot{x}_n \qquad (2\text{-}17)$$

where

$$A_{30} = \frac{\Omega^2}{\omega^2 mD} \qquad (2\text{-}18)$$

$$A_{31} = -\frac{\Omega^2}{D} \qquad (2\text{-}19)$$

$$A_{32} = -\frac{\theta\Omega^2}{D} \qquad (2\text{-}20)$$

$$A_{33} = 1 - \frac{(1+\frac{1}{2}\theta^2\Omega^2)}{D} \qquad (2\text{-}21)$$

or in matrix form

$$\begin{bmatrix} \Delta t^2\ddot{x}_{n+1} \\ \Delta t\dot{x}_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} A_{33} & A_{32} & A_{31} \\ [1+\gamma(A_{31}-1)] & (1+\gamma A_{32}) & \gamma A_{31} \\ [\frac{1}{2}+\beta(A_{33}-1)] & (1+\beta A_{32}) & (1+\beta A_{31}) \end{bmatrix} \begin{bmatrix} \Delta t^2\ddot{x}_n \\ \Delta t\dot{x}_n \\ x_n \end{bmatrix} + \begin{bmatrix} A_{30} f_{n+\theta} \\ \gamma A_{30} f_{n+\theta} \\ \beta A_{30} f_{n+\theta} \end{bmatrix} \qquad (2\text{-}2$$

$$\begin{bmatrix} \Delta t^2 \ddot{x}_{n+1} \\ \Delta t \dot{x}_{n+1} \\ x_{n+1} \end{bmatrix} = \underset{\sim}{A} \begin{bmatrix} \Delta t^2 \ddot{x}_{n} \\ \Delta t \dot{x}_{n} \\ x_{n} \end{bmatrix} + \begin{bmatrix} A_{30} \\ \gamma A_{30} \\ \beta A_{30} \end{bmatrix} f_{n+\theta} \tag{2-23}$$

or

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}\, \underset{\sim}{X}_n + \underset{\sim}{L}\, f_{n+\theta} \tag{2-24}$$

with the abbreviations

$$\begin{aligned} \omega\Delta t &= \Omega \\ \omega^2 &= \frac{k}{m} \\ D &= \theta(1+\beta\theta^2\Omega^2) \\ f_{n+\theta} &= (1-\theta)f_n + \theta f_{n+1} \end{aligned} \tag{2-25}$$

The eigenvalues of $\underset{\sim}{A}$ [Equation (2-22)] satisfy the following characteristic equation in λ, originating from

$$\det \mid \underset{\sim}{A} - \lambda \underset{\sim}{I} \mid = 0$$

or (2-26)

$$\lambda^3 - 2A_1\,\lambda^2 + A_2\lambda - A_3 = 0$$

where

$$\begin{aligned} 2A_1 &= A_{33} + [1+\gamma A_{32}] + [1+\beta A_{31}] \\ &= 2 + \beta A_{31} + \gamma A_{32} + A_{33} \end{aligned} \tag{2-27}$$

$$\begin{aligned} A_2 &= [A_{33}+A_{32}(\gamma-1)] + [A_{33}+A_{31}(\beta-\tfrac{1}{2})] + [1+\beta A_{31}+\gamma(A_{32}-A_{31})] \\ &= 1 + (2\beta-\tfrac{1}{2}-\gamma)A_{31} + (2\gamma-1)A_{32} + 2A_{33} \end{aligned} \tag{2-28}$$

$$\begin{aligned} A_3 &= (1+\beta A_{31})[A_{33}+(\gamma-1)A_{32}] + (1+\beta A_{32})A_{31}(1-\gamma) - A_{31}[\tfrac{1}{2}+\beta(A_{33}-1)] \\ &= A_{33} + (\gamma-1)A_{32} + A_{31}\,[\beta+\tfrac{1}{2}-\gamma] \end{aligned} \tag{2-29}$$

If $A_3 = 0$, Equation (2-26) reduces to a quadratic, and the algorithm is of the Newmark-β type.

Replacing A_{31}, A_{32}, and A_{33} from Equations (2-24) in (2-29), we obtain

$$A_3 = \frac{(\theta-1)\left\{1+\Omega^2[\beta(\theta^2+\theta+1) - \tfrac{1}{2}(\theta-1)-\gamma]\right\}}{D} \tag{2-30}$$

Furthermore, the difference Equation (2-23) takes at steps n and n+1 yields

$$\underset{\sim}{X}_n = \underset{\sim}{A}\,\underset{\sim}{X}_{n-1} + \underset{\sim}{L}\,f_{n+\theta} \tag{2-31}$$

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}\,\underset{\sim}{X}_n + \underset{\sim}{L}\,f_{n+\theta+1} \tag{2-32}$$

Combining Equations (2-31) and (2-32), we obtain the following recurrence relationship

$$\begin{aligned}\underset{\sim}{X}_{n+1} &= \underset{\sim}{A}[\underset{\sim}{A}\,\underset{\sim}{X}_{n-1} + \underset{\sim}{L}\,f_{n+\theta}] + \underset{\sim}{L}\,f_{n+\theta+1}\\ &= \underset{\sim}{A}^2\underset{\sim}{X}_{n-1} + \underset{\sim}{A}\,\underset{\sim}{L}\,f_{n+\theta} + \underset{\sim}{L}\,f_{n+\theta+1}\end{aligned} \tag{2-33}$$

Successive repeats produce the result

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}^{n+1}\underset{\sim}{X}_o + \sum_{k=o}^{k=n} \underset{\sim}{A}^k\underset{\sim}{L}f_{n-k+\theta+1} \tag{2-34}$$

We observe that for no external loading (f = 0) repetition of Equations (2-32) and (2-31) yield

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}\underset{\sim}{X}_n = \underset{\sim}{A}^2\underset{\sim}{X}_{n-1} = \underset{\sim}{A}^3\underset{\sim}{X}_{n-2} \tag{2-35}$$

However, by the Cayley-Hamilton theorem on matrices (Reference 46), the matrix A satisfies its own characteristic equation [Equation (2-25)], i.e.

$$\underset{\sim}{A}^3 - 2A_1\underset{\sim}{A}^2 + A_2\underset{\sim}{A} - A_3\underset{\sim}{I} = \underset{\sim}{0} \tag{2-36}$$

Substituting for $\underset{\sim}{A}^3$ from Equation (2-36) in (2-35) we obtain

$$\begin{aligned}\underset{\sim}{X}_{n+1} &= 2A_1\underset{\sim}{A}^2 - A_2\underset{\sim}{A} + \underset{\sim}{A}_3\,\underset{\sim}{X}_{n-2}\\ &= 2A_1\underset{\sim}{A}^2\underset{\sim}{X}_{n-2} - A_2\underset{\sim}{A}\,\underset{\sim}{X}_{n-2} + A_3\underset{\sim}{X}_{n-2}\end{aligned} \tag{2-37}$$

Noting that

$$\underset{\sim}{X}_n = \underset{\sim}{A}^2\underset{\sim}{X}_{n-2} \tag{2-38}$$

$$\underset{\sim}{X}_{n-1} = \underset{\sim}{A}\,\underset{\sim}{X}_{n-2} \tag{2-39}$$

we finally obtain

$$\underset{\sim}{X}_{n+1} = 2A_1\underset{\sim}{X}_n - A_2\underset{\sim}{X}_{n-1} + A_3\underset{\sim}{X}_{n-2} \tag{2-40}$$

Equation (2-40) is the general recurrence relationship of the linear undamped homogeneous equation of (2-1) (with f = 0) for the state vector $\underset{\sim}{X}$. It is, therefore, valid for any of its components such as the displacements x_n.

$$x_{n+1} = 2A_1x_n - A_2x_{n-1} + A_3x_{n-2} \tag{2-41}$$

Omitting a large portion of the analysis which can be found in Reference 15, 16, 17, or 47, Hilber and Hughes concluded that, for unconditional stability, we must have

$$\gamma = \tfrac{1}{2} \tag{2-42}$$

$$\theta \geq 1 \tag{2-43}$$

and

$$\frac{\theta}{2(\theta+1)} \geq \beta \geq \frac{2\theta^2-1}{4(2\theta^3-1)} \tag{2-44}$$

Hilber and Hughes found that when $\theta \simeq 1.366025$, the higher modes are not damped out. At about $\theta = 1.420815$, they found that the maximum damping of high frequencies takes place. They also found that relative period error increases as β decreases from 0.25 on down, and θ increases from 1 upward, simultaneously.

To improve the dissipation properties, they developed the so called Hilber-Hughes α-method (References 15, 16, and 17) which is summarized in Tables 1-1 and 1-2.

CHAPTER 3

ANALYSIS OF HILBER-HUGHES COLLOCATION METHOD

From Equation (2-6) we solve for the acceleration at collocation point $n+\theta$

$$\ddot{\mathbf{x}}_{n+\theta} = \frac{1}{\beta\theta^2\Delta t^2}\,\mathbf{x}_{n+\theta} - \frac{1}{\beta\theta^2\Delta t^2}\,\mathbf{x}_n - \frac{1}{\beta\theta\Delta t}\,\dot{\mathbf{x}}_n - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{x}}_n \tag{3-1}$$

Substituting Equation (3-1) into the next equation, i.e. Equation (3-2)

$$\dot{\mathbf{x}}_{n+\theta} = \dot{\mathbf{x}}_n + \theta\Delta t[(1-\gamma)\ddot{\mathbf{x}}_n+\ddot{\mathbf{x}}_{n+\theta}] \tag{3-2}$$

the expression for velocity at the point of collocation follows in terms of historical variables and $x_{n+\theta}$

$$\dot{\mathbf{x}}_{n+\theta} = \frac{\gamma}{\beta\theta\Delta t}\left(\mathbf{x}_{n+\theta}-\mathbf{x}_n\right) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{x}}_n + \theta\Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{x}}_n \tag{3-3}$$

Hence, replacing these in the equations of motion

$$\mathbf{M}\,\ddot{\mathbf{x}}_{n+\theta} + \mathbf{C}\,\dot{\mathbf{x}}_{n+\theta} + \mathbf{K}\,\mathbf{x}_{n+\theta} = \mathbf{F}_{n+\theta} \tag{3-4}$$

we obtain

$$\begin{aligned}
&\left[\frac{1}{\beta\theta^2\Delta t^2}\mathbf{M} + \frac{\gamma}{\beta\theta\Delta t}\mathbf{C} + \mathbf{K}\right]\mathbf{x}_{n+\theta} \\
&= \mathbf{F}_{n+\theta} + \mathbf{M}\left[\frac{1}{\beta\theta^2\Delta t^2}\mathbf{x}_n + \frac{1}{\beta\theta\Delta t}\dot{\mathbf{x}}_n + \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{x}}_n\right] \\
&\quad + \mathbf{C}\left[\frac{\gamma}{\beta\theta\Delta t}\mathbf{x}_n + \left(\frac{\gamma}{\beta} - 1\right)\dot{\mathbf{x}}_n + \left(\frac{\gamma}{2\beta} - 1\right)\theta\Delta t\;\ddot{\mathbf{x}}_n\right]
\end{aligned} \tag{3-5}$$

Solve for the displacement vector $\underset{\sim}{x}_{n+\theta}$ at the collocation point $n+\theta$ in terms of all the historical variables. Next, solve Equation (2-5) for $\ddot{\underset{\sim}{x}}_{n+1}$ in terms of $\ddot{\underset{\sim}{x}}_n$, $\ddot{\underset{\sim}{x}}_{n+\theta}$. Then substitute $\ddot{x}_{n+\theta}$ from Equation (3-1) to obtain the acceleration at next time step n+1

$$\ddot{x}_{n+1} = \frac{1}{\theta}\ddot{\underset{\sim}{x}}_{n+\theta} - \left(\frac{1}{\theta} - 1\right)\ddot{\underset{\sim}{x}}_n$$

$$= \frac{1}{\beta\theta^3\Delta t^2}(\underset{\sim}{x}_{n+\theta} - \underset{\sim}{x}_n) - \frac{1}{\beta\theta^2\Delta t}\dot{\underset{\sim}{x}}_n - \left(\frac{1}{2\beta\theta} - 1\right)\ddot{\underset{\sim}{x}}_n \qquad (3\text{-}6)$$

Now, from Equation (2-9) $\dot{\underset{\sim}{x}}_{n+1}$ is

$$\dot{\underset{\sim}{x}}_{n+1} = \dot{\underset{\sim}{x}}_n + \Delta t[(1-\gamma)\ddot{\underset{\sim}{x}}_n + \gamma\ddot{\underset{\sim}{x}}_{n+1}]$$

and replacing the value of $\underset{\sim}{x}_{n+1}$ from Equation (3-6), we finally arrive at

$$\dot{\underset{\sim}{x}}_{n+1} = \frac{\gamma}{\beta\theta^3\Delta t}\left(\underset{\sim}{x}_{n+\theta} - \underset{\sim}{x}_n\right) + \left(1 - \frac{\gamma}{\beta\theta^2\Delta t}\right)\dot{\underset{\sim}{x}}_n$$

$$+ \left(1 - \frac{\gamma}{2\beta\theta}\right)\Delta t\ \ddot{\underset{\sim}{x}}_n \qquad (3\text{-}7)$$

Furthermore, the displacements at time step n+1 can be obtained from

$$\underset{\sim}{x}_{n+1} = \underset{\sim}{x}_n + \Delta t\ \dot{\underset{\sim}{x}}_n + \Delta t^2\left[\left(\frac{1}{2}-\beta\right)\ \ddot{\underset{\sim}{x}}_n + \beta\ \ddot{\underset{\sim}{x}}_{n+1}\right]$$

$$= \frac{1}{\theta^3}\underset{\sim}{x}_{n+\theta} + \left(1 - \frac{1}{\theta^3}\right)\underset{\sim}{x}_n + \Delta t\left(1 - \frac{1}{\theta^2}\right)\dot{\underset{\sim}{x}}_n + \Delta t^2\left(\frac{1}{2} - \frac{1}{2\theta}\right)\ddot{\underset{\sim}{x}}_n \qquad (3\text{-}8)$$

CHAPTER 4

SUMMARY OF HILBER-HUGHES ALGORITHM

The following steps describe the algorithmic flow. Initial conditions on some of $\mathbf{x}_0$, $\dot{\mathbf{x}}_0$, $\ddot{\mathbf{x}}_0$, $\mathbf{F}_0$ are given. (See Chapter 2.) Also effective matrices are factorized by a Cholesky decomposition.

1. Obtain $\mathbf{x}_{n+\theta}$

$$\mathbf{x}_{n+\theta} = \frac{1}{\beta\theta^2\Delta t^2}\mathbf{M} + \frac{\gamma}{\beta\theta\Delta t}\mathbf{C} + \mathbf{K}^{-1} \; \mathbf{F}_{n+\theta}$$

$$+ \mathbf{M}\; \frac{1}{\beta\theta^2\Delta t^2}\mathbf{x}_n + \frac{1}{\beta\theta\Delta t}\dot{\mathbf{x}}_n + \frac{1}{2\beta} - 1 \; \ddot{\mathbf{x}}_n$$

$$+ \mathbf{C}\; \frac{\gamma}{\beta\theta\Delta t}\mathbf{x}_n + \frac{\gamma}{\beta} - 1 \; \dot{\mathbf{x}}_n + \frac{\gamma}{2\beta} - 1 \; \theta\Delta t \; \ddot{\mathbf{x}}_n \tag{4-1}$$

2. Obtain $\ddot{\mathbf{x}}_{n+1}$

$$\ddot{\mathbf{x}}_{n+1} = \frac{1}{\beta\theta^3\Delta t^2}\; \mathbf{x}_{n+\theta} - \mathbf{x}_n \; - \frac{1}{\beta\theta^2\Delta t}\dot{\mathbf{x}}_n - \frac{1}{2\beta\theta} - 1 \; \ddot{\mathbf{x}}_n \tag{4-2}$$

3. Obtain $\dot{\mathbf{x}}_{n+1}$

$$\dot{\mathbf{x}}_{n+1} = \frac{\gamma}{\beta\theta^3\Delta t}\; \mathbf{x}_{n+\theta} - \mathbf{x}_n \; + \; 1 - \frac{\gamma}{\beta\theta^2\Delta t}\; \dot{\mathbf{x}}_n + 1 - \frac{\gamma}{2\beta\theta}\; \Delta t \; \ddot{\mathbf{x}}_n \tag{4-3}$$

4. Obtain $\mathbf{x}_{n+1}$

$$\mathbf{x}_{n+1} = \frac{1}{\theta^3}\mathbf{x}_{n+\theta} + 1 - \frac{1}{\theta^3}\; \mathbf{x}_n + \Delta t \; 1 - \frac{1}{\theta^2}\; \dot{\mathbf{x}}_n + \Delta t^2 \; \frac{1}{2} - \frac{1}{2\theta}\; \ddot{\mathbf{x}}_n \tag{4-4}$$

Repeat steps 1-4 for all time steps.

CHAPTER 5

MODIFIED HILBER-HUGHES METHOD

Motivation for the extension outlined below comes from the need to improve upon and better control the dissipation and period elongation characteristics of the algorithm, while retaining some degree of simplicity. As can be seen by studying the equations of equilibrium (Reference 1), implicit methods require inversions of large matrices that considerably slow the solution process.

This scheme is based on ideas proposed by Hilber and Hughes (References 15, 16, and 17). The main difference compared with the previous method by Hilber et al is that the equilibrium equations are satisfied at the collocation point $n+\theta$

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{x}}_{n+\theta} + \underset{\sim}{C}\,\dot{\underset{\sim}{x}}_{n+\theta} + (1+\alpha)\underset{\sim}{K}\,\underset{\sim}{x}_{n+\theta} - \alpha\underset{\sim}{K}\,\underset{\sim}{x}_n = \underset{\sim}{F}_{n+\theta} \tag{5-1}$$

where the two Hilber-Hughes methods appear combined. As in the previous chapter it leads to a displacement vector at time $n+\theta$ given by

$$\begin{aligned}\underset{\sim}{x}_{n+\theta} + &\left[\frac{1}{\beta\theta^2\Delta t^2}\underset{\sim}{M} + \frac{\gamma}{\beta\theta\Delta t}\underset{\sim}{C} + (1+\alpha)\underset{\sim}{K}\right]^{-1}\Bigg[\underset{\sim}{F}_{n+\theta} + \\ &+ \underset{\sim}{M}\left\{\frac{1}{\beta\theta^2\Delta t^2}\underset{\sim}{x}_n + \frac{1}{\beta\theta\Delta t}\dot{\underset{\sim}{x}}_n + \left(\frac{1}{2\beta} - 1\right)\ddot{\underset{\sim}{x}}_n\right\} + \\ &+ C\left\{\frac{\gamma}{\beta\theta\Delta t}\underset{\sim}{x}_n + \left(\frac{\gamma}{\beta} - 1\right)\dot{\underset{\sim}{x}}_n + \left(\frac{\gamma}{2\beta} - 1\right)\theta\Delta t\ \ddot{\underset{\sim}{x}}_n\right\} + \\ &+ \alpha\underset{\sim}{K}\,\underset{\sim}{x}_n\Bigg]\end{aligned} \tag{5-2}$$

In this method the following approximations are used for the acceleration, velocity, and displacement

$$\begin{aligned}\ddot{x}_{n+\theta} &= (1-\theta)\ddot{x}_n + \theta\ \ddot{x}_{n+1} \\ \dot{x}_{n+\theta} &= \dot{x}_n + \theta\Delta t\left[(1-\gamma)\ddot{x}_n + \gamma\ \ddot{x}_{n+\theta}\right] \\ x_{n+\theta} &= x_n + \theta\Delta t\ \dot{x}_n \\ &\quad + \theta^2\Delta t^2\left[\left(\frac{1}{2} - \beta\right)\ddot{x}_n + \beta\ \ddot{x}_{n+\theta}\right]\end{aligned} \tag{5-3}$$

or the equivalent set

$$\ddot{x}_{n+\theta} = (1-\theta)\ddot{x}_n + \theta\ \ddot{x}_{n+1}$$

$$\dot{x}_{n+\theta} = \dot{x}_n + \theta\Delta t\left[(1-\gamma\theta)\ddot{x}_n + \gamma\theta\ \ddot{x}_{n+1}\right]$$

$$x_{n+\theta} = x_n + \theta\Delta t\ \dot{x}_n + \qquad$$

$$+ \theta^2\Delta t^2\left[(\tfrac{1}{2} - \beta\theta)\ddot{x}_n + \beta\theta\ \ddot{x}_{n+1}\right] \qquad (5\text{-}4)$$

The stability characteristics have been analyzed and documented in detail in Appendix A.

Table 5-1 displays the $\underset{\sim}{A}$ amplification matrix for the Newmark β method (References 1 and 47), while Tables 5-2 and 5-3 display the $\underset{\sim}{A}$ matrix for the Wilson θ method.

To further the discussion and tie in the results of the stability analysis with those of the Wilson-θ and Newmark-β methods, we observe the following:

The elements of the amplification matrix for the α-collocation Hughes operator assume the Wilson form (Table 5-3), as can be seen below, when

$$\alpha = 0,\ \gamma = \tfrac{1}{2},\ \beta = \frac{1}{6} \text{ becomes } B = \frac{24\pi^2 p^2}{\theta\left[6+12\pi\theta\xi p+4\pi^2\theta^2 p^2\right]}\ . \qquad (5\text{-}5)$$

Let's denote the elements of the $\underset{\sim}{A}$ matrix by a_{ij}.

First, note that the first element of the $\underset{\sim}{A}$ matrix from Table 5-2 is

$$a_{11} = \left(\frac{\theta}{6} - \frac{1}{2}\right)\theta^2 B - 2\left(1 - \frac{\theta}{2}\right)\theta\kappa - (1-\theta)\frac{B}{\omega^2\Delta t^2} \qquad (5\text{-}6)$$

But since

$$\frac{\theta^3 B}{6} = \frac{4\pi^2\theta^2 p^2}{6+12\pi\theta\xi p+4\pi^2 p^2\theta^2}$$

$$\theta^2\kappa = \frac{12\pi\xi\theta p}{\left[6+12\pi\theta\xi p+4\pi^2 p^2\theta^2\right]} \qquad (5\text{-}7)$$

and

$$\frac{B\theta}{\omega^2\Delta t^2} = \frac{6}{\left[6+12\pi\theta\xi p+4\pi^2 p^2\theta^2\right]} \qquad (5\text{-}8)$$

or

$$\frac{\theta^3 B}{6} + \theta^2 \kappa + \frac{B\theta}{\omega^2 \Delta t^2} = 1 \tag{5-9}$$

it becomes

$$a_{11} = 1 - \frac{1}{2}\theta^2 B - 2\theta\kappa - \frac{B}{\omega^2 \Delta t^2} \tag{5-10}$$

and its equivalent form is given in Table 5-3. Furthermore,

$$a_{12} = -(2\kappa + \theta B) \tag{5-11}$$

$$a_{13} = -B \tag{5-12}$$

$$a_{22} = 1 - \kappa - \frac{1}{2}\theta B \tag{5-13}$$

$$a_{23} = -\frac{1}{2}B \tag{5-14}$$

$$a_{32} = 1 - \frac{1}{6}B\theta - \frac{1}{3}\kappa \tag{5-15}$$

$$a_{33} = 1 - \frac{1}{6}B \tag{5-16}$$

Again from Table 5-2

$$a_{21} = -\frac{(1-\theta)B}{2\omega^2 \Delta t^2} - 2\left(1 - \frac{1}{2}\theta\right)\frac{1}{2}\theta\kappa + \frac{1}{2}\left(\frac{\theta}{6} - \frac{1}{2}\right)\theta^2 B = \tag{5-17}$$

$$= \frac{1}{2} - \frac{\theta^2}{4}B - \theta\kappa - \frac{B}{2\omega^2 \Delta t^2} + \frac{\theta B}{2\omega^2 \Delta t^2} + \frac{\theta^2 \kappa}{2} + \frac{1}{12}\theta^3 B \tag{5-18}$$

We use the equality proven above, i.e.

$$\frac{\theta B}{2\omega^2 \Delta t^2} + \frac{\theta^2 \kappa}{2} + \frac{1}{12}\theta^3 B = \frac{1}{2} \tag{5-19}$$

and we arrive at

$$a_{21} = 1 - \frac{1}{4}\theta^2 B - \kappa\theta - \frac{B}{2\omega^2 \Delta t^2} \tag{5-20}$$

which is the equivalent form of Table 5-2, i.e. Table 5-3.

Similarly,

$$a_{31} = \left(\frac{1}{2} - \frac{1}{6}\right) - \frac{(1-\theta)}{6\omega^2 \Delta t^2} B - 2\left(1 - \frac{1}{2}\theta\right)\frac{\theta}{6}\kappa +$$

$$\left(\frac{1}{6}\theta - \frac{1}{2}\right)\theta^2 \frac{B}{6} = \frac{1}{2} - \frac{1}{12}B\theta^2 - \frac{1}{3}\kappa\theta - \frac{B}{6\omega^2 \Delta t^2} \tag{5-21}$$

or given by Table 5-3. In these equations, the parameter B basically assumes the Wilson form as given in Table 5-2.

Setting $\alpha = 0$, $\theta = 1$, we recover the Newmark-β amplification matrix (Table 5-1) by a similar reduction process.

We further observe that at $\theta = 1$, the Wilson amplification matrix $\underset{\sim}{A}$ becomes

$$\underset{\sim}{A} = \left[\begin{array}{c|c|c} -\frac{1}{3}B - \kappa & -(2\kappa + B) & -B \\ \hline \frac{1}{2} - \frac{1}{2}\kappa - \frac{1}{6}B & 1 - (\kappa + \frac{1}{2}B) & -\frac{1}{2}B \\ \hline \frac{1}{3} - \frac{1}{6}\kappa - \frac{1}{18}B & 1 - \frac{1}{6}B - \frac{1}{3}\kappa & 1 - \frac{1}{6}B \end{array}\right] \tag{5-22}$$

with $$B = \frac{24\pi^2 p^2}{[6 + 12\pi\xi p + 4\pi^2 p^2]} . \tag{5-23}$$

It agrees with the α-collocation of Hughes only if $\gamma = \frac{\theta}{2}$, $\beta = \frac{\theta}{6}$, $\alpha = 0$, and $\theta = 1$ (i.e., $\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$, $\alpha = 0$, $\theta = 1$). It is known that at $\theta = 1$ the Wilson method is unstable.

In the present study, the stability range (for $\xi \geq 0$) is summarized on pages A-14 (Case $A_3 \neq 0$) and A-21 (Case $A_3 = 0$) of the Appendix. Definition of the parameter A_3 can be found in the Appendix.

Case $A_3 = 0$ was the only case analyzed previously.[15,16,17] The case analyzed corresponding to $A_3 = 0$, for which the stability range (with $\xi = 0$) given was:

$$\gamma = \frac{1}{2}$$

$\alpha = 0$ (There is no α in the Hilber-Hughes collocation algorithm; see Tables 1-1 and 1-2.)

$$\theta \geq 1 \tag{5-24}$$

and

$$\frac{\theta}{2(\theta+1)} \geq \beta \geq \frac{2\theta^2-1}{4(2\theta^3-1)} \tag{5-25}$$

In the case of the α-method, the given range was:

$$\gamma = \frac{1}{2} - \alpha$$

$$\beta = \frac{1}{4}(1-\alpha)^2$$

$$-\frac{1}{3} \leq \alpha \leq 0 \tag{5-26}$$

(set θ = 1 in the modified form)

Hence it should be obvious that by including the general case $A_3 \neq 0$ in the present study, the stability range has been broadened and the two methods of Hilber-Hughes combined in a single one.

TABLE 5-1. NEWMARK β $\underset{\sim}{A}$ AMPLIFICATION MATRIX

$$\begin{bmatrix} \Delta t^2 \ddot{\chi}_{t+1} \\ \Delta t \dot{\chi}_{t+1} \\ \chi_{t+1} \end{bmatrix} = \begin{bmatrix} \left[\left(\beta - \frac{1}{2}\right) B - 2(1-\gamma) x\right] & -(2x + B) & -B \\ \left[(1-\gamma) - \left(\frac{1}{2} - \beta\right)\gamma B - 2(1-\gamma)\gamma x\right] & (1 - \gamma B - 2x\gamma) & -\gamma B \\ \left[\left(\frac{1}{2} - \beta\right) - \left(\frac{1}{2} - \beta\right)\beta B - 2(1-\gamma)\beta x\right] & (1 - \beta B - 2x\beta) & (1 - \beta B) \end{bmatrix} \begin{bmatrix} \Delta t^2 \ddot{\chi}_t \\ \Delta t \dot{\chi}_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} \frac{B}{\omega^2} f_{t+1} \\ \frac{\gamma B}{\omega^2} f_{t+1} \\ \frac{\beta B}{\omega^2} f_{t+1} \end{bmatrix}$$

ABBREVIATIONS:

$$B = \frac{4\pi^2 p^2}{[1 + 4\pi\gamma\xi p + 4\pi^2 \beta p^2]}$$

$$\xi = 2\pi x \frac{p}{B} = \frac{x\omega\Delta t}{B}$$

$$\omega = \frac{2\pi}{T}$$

$$p = \frac{\Delta t}{T}$$

$$\begin{bmatrix} \Delta t^2 \ddot{\chi}_{t+1} \\ \Delta t \dot{\chi}_{t+1} \\ \chi_{t+1} \end{bmatrix} = \underset{\sim}{A} \begin{bmatrix} \Delta t^2 \ddot{\chi}_t \\ \Delta t \dot{\chi}_t \\ \chi_t \end{bmatrix} + \underset{\sim}{L}$$

TABLE 5-2. WILSON Θ $\underset{\sim}{A}$ AMPLIFICATION MATRIX

$$\begin{bmatrix} \Delta t^2 \ddot{\chi}_{t+1} \\ \Delta t\, \dot{\chi}_{t+1} \\ \chi_{t+1} \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{1}{\Theta} x\Theta - \frac{1}{3} B\Theta^2\right) & -(2x + B\Theta) & -B \\ \left(1 - \frac{1}{2\Theta} - \frac{x\Theta}{2} - \frac{1}{6} B\Theta^2\right) & \left(1 - x - \frac{1}{2} B\Theta\right) & -\frac{1}{2} B \\ \left(\frac{1}{2} - \frac{1}{6\Theta} - \frac{x\Theta}{6} - \frac{B\Theta^2}{18}\right) & \left(1 - \frac{1}{6} B\Theta - \frac{x}{3}\right) & \left(1 - \frac{1}{6} B\right) \end{bmatrix} \begin{bmatrix} \Delta t^2 \ddot{\chi}_t \\ \Delta t \dot{\chi}_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} \frac{B}{\omega^2} f_{t+\Theta\Delta t} \\ \frac{B}{2\omega^2} f_{t+\Theta\Delta t} \\ \frac{B}{6\omega^2} f_{t+\Theta\Delta t} \end{bmatrix}$$

ABBREVIATIONS:

$$B = \frac{24\pi^2 p^2}{[6\Theta + 12\pi\xi\Theta^2 p + 4\pi^2\Theta^3 p^2]}$$

$$p = \frac{\Delta t}{T}$$

$$x = \frac{\xi B}{2\pi p}$$

TABLE 5-3. ALTERNATE FORM OF WILSON Θ $\underset{\sim}{A}$ AMPLIFICATION MATRIX

$$
\begin{bmatrix} \Delta t^2 \ddot{\chi}_{t+1} \\ \Delta t \dot{\chi}_{t+1} \\ \chi_{t+1} \end{bmatrix}
=
\left[\begin{array}{c|c|c}
\left(1 - \frac{1}{2} B\Theta^2 - 2x\Theta - \frac{B}{\omega^2 \Delta t^2}\right) & -(2x + B\Theta) & -B \\
\left(1 - \frac{1}{4} B\Theta^2 - x\Theta - \frac{B}{2\omega^2 \Delta t^2}\right) & \left((1 - x - \frac{1}{2} B\Theta)\right) & -\frac{1}{2} B \\
\left(\frac{1}{2} - \frac{1}{12} B\Theta^2 - \frac{1}{3} x\Theta - \frac{B}{6\omega^2 \Delta t^2}\right) & \left(1 - \frac{1}{6} B\Theta - \frac{x}{3} B\right) & \left(1 - \frac{1}{6} B\right)
\end{array}\right]
\begin{bmatrix} \Delta t^2 \ddot{\chi}_t \\ \Delta t \dot{\chi}_t \\ \chi_t \end{bmatrix}
+
\begin{bmatrix} \frac{B}{\omega^2} f_{t+\theta\Delta t} \\ \frac{B}{2\omega^2} f_{t+\theta\Delta t} \\ \frac{B}{6\omega^2} f_{t+\theta\Delta t} \end{bmatrix}
$$

CHAPTER 6

DISCUSSION AND CONCLUSIONS

Chapter 1 is an introduction to some of the methods used in structural dynamics. Chapter 2 deals with the stability of the Hilber-Hughes collocation method and Chapters 3 and 4 indicate a way to implement the procedure, while Chapter 5 offers a possible extension of the Hilber-Hughes algorithm. Furthermore, Chapter 5 discusses how the amplification matrix $\underset{\sim}{A}$ degenerates to the Newmark β amplification matrix, etc. for appropriate choice of the parameters. Appendix A deals exclusively with the analysis of the stability characteristics of the modified Hilber-Hughes method.

In References 15 and 17 Hilber and Hughes indicated that such methods may tend to overshoot the actual solution at the very few initial steps. For this reason,plots of the spectral norm of the amplification matrix $\underset{\sim}{A}$ (see Reference 46 on how to calculate $||\underset{\sim}{A}||$) in addition to its spectral radius have been included in all cases. These data could be used in future studies of algorithmic performance. For more details see References 15 and 17.

This section, therefore serves the purpose of a brief discussion of the obtained results and how they can be used in practice.

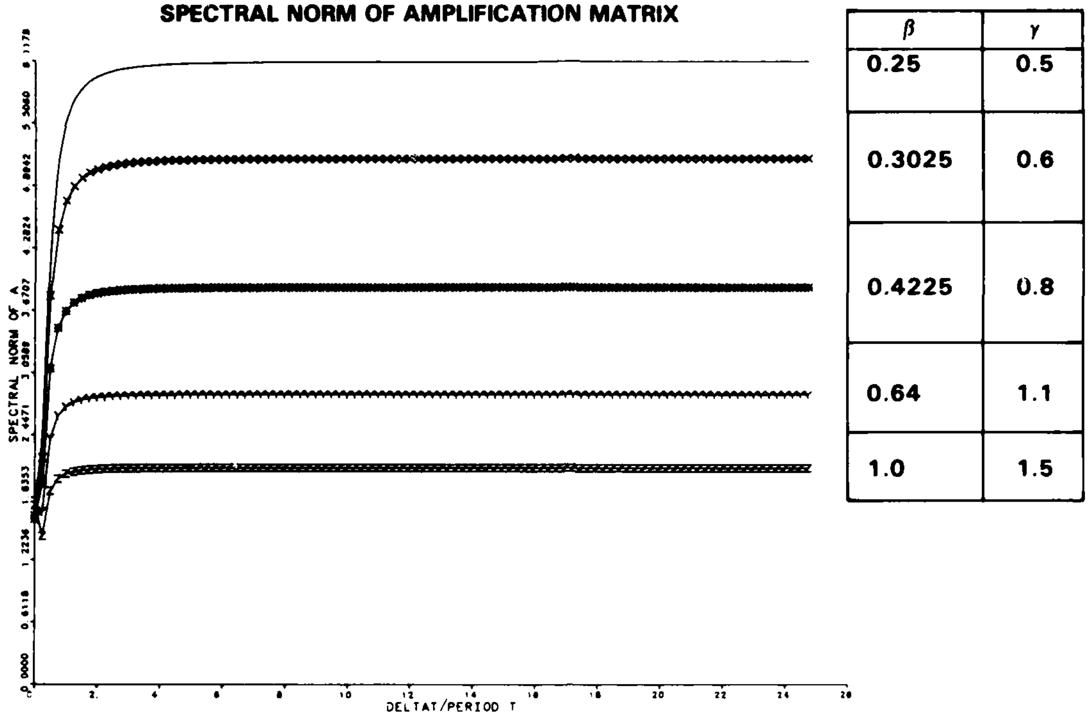
Table 6-1 relates the figures and all tables (6-2 - 6-19) from which all figures of this Chapter were obtained. For example Tables 6-2 to 6-6 pertain to Figures 6-1 to 6-3. Table 6-9 is used in relation to Figure 6-22. All the other tables can be connected to the relevant figures by consulting Table 6-1.

Although this report deals with the modified Hilber-Hughes algorithm, it is beneficial to have some visual feel with some of the other algorithms. Figures 6-1(a) to 6-3(a) give the spectral norm and radius of the amplification matrix $\underset{\sim}{A}$ by the Newmark β method for the case of no structural damping (ξ = 0) for various values of β and γ respectively and over a wide range of frequencies (see abscissa of plots). Figures 6-1(b) to 6-3(b) give the spectral norm and radius for the same parameters for the Wilson θ method. Figures 6-4(a) and 6-5(a) are plots of spectral radius for the Newmark β method for various β, γ combinations and for some structural damping (ξ = 0.10) over a frequency range. The same is true for Figures 6-4(b) and 6-5(b) for the Wilson θ method. Figures 6-6(a) and 6-6(b) display the spectral norm for the Hughes amplification matrix $\underset{\sim}{A}$ (θ = 1.4, ξ = 0.10, γ = 0.10, β = 0.916275) and the Wilson θ method (θ = 1.4, ξ = 0.10) respectively. Figures 6-7(a) and (b) are plots of spectral norm and radius by the Newmark β method for a choice of β, γ and structural damping (ξ = 0.10). Figures 6-8(a), 6-8(b), 6-9(a) and 6-9(b) display spectral radius variation for the Wilson θ, Hughes and Newmark β methods. All the remaining figures (Figures 6-9 to 6-22) pertain to the modified Hilber-Hughes method. Figure 6-10 displays spectral norm (a) and spectral radius (b) for α = -0.10, while Figures 6.11 to 6-17 are valid for α = 0.0. Figures 6-18 and 6-19 are for the case of α = -0.10, while Figures 6-20 to 6-22 for α = -0.20.

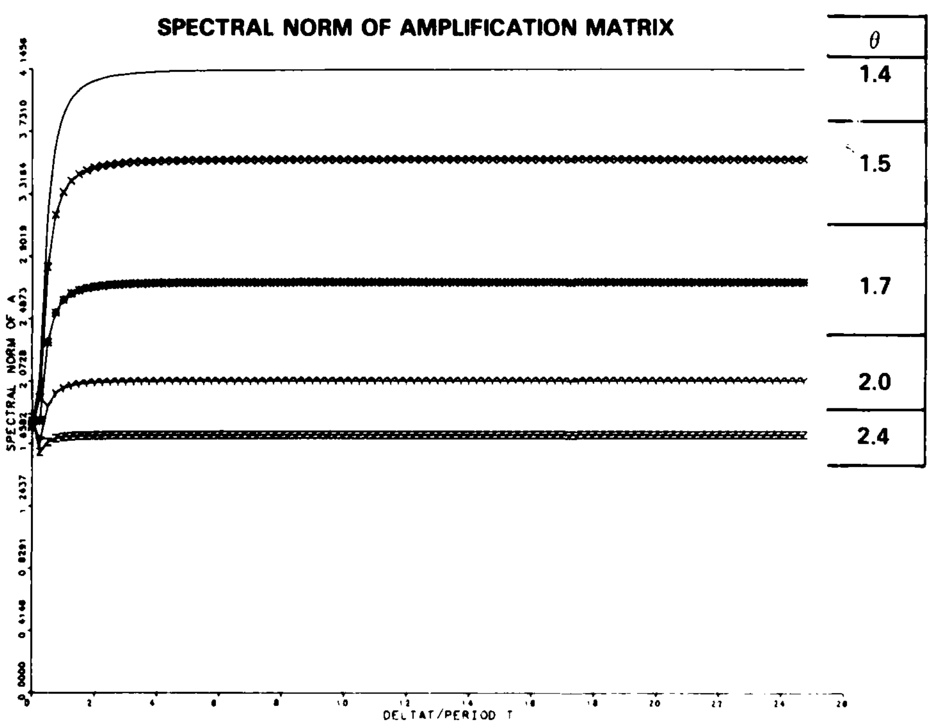
The usefulness of these figures and tables can be explained as follows. Consider the case of $\alpha = 0.0$, $\gamma = 1.0$, $\theta = 1.0$, $\xi = 0.10$. Figure 6-12(b) indicates that for the case of $\beta = 0.50$, the spectral radius drops and then increases to nearly 1.000 in the high frequency range. The same applies to the spectral norm. Table 6-9 indicates this and, depending on what type of high frequency dissipation may be needed, we can see that $\beta = 1.0$ provides this more effectively than $\beta = 0.50$, $\beta = 1.50$, and $\beta = 2.0$. At the same time, the spectral norm is higher than for the case of $\beta = 2.0$.

Figures 6-10 through 6-22 and Tables 6-7 through 6-19, therefore, give reasonable estimates of the parameters α, β, γ, and θ so that the linearized structural system is unconditionally stable. Otherwise, one may have to resort to Tables A-1 and A-2 of Appendix A in order to obtain these quantities.

In conclusion, therefore, the present extension can be used as a starting point for a more detailed study of this algorithm. It should be implemented in a finite element program in order to verify its superior or inferior performance with respect to dissipation, overshooting, and phase shift properties for problems with known solutions.



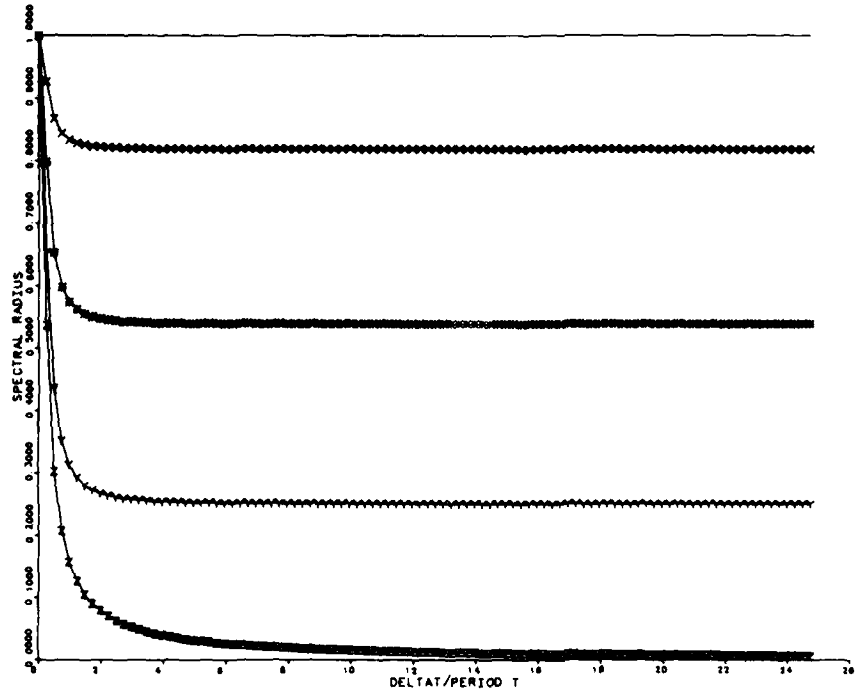
(A) BY NEWMARK β METHOD FOR $\xi = 0$



(B) BY WILSON θ METHOD FOR $\xi = 0$

FIGURE 6-1. SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$

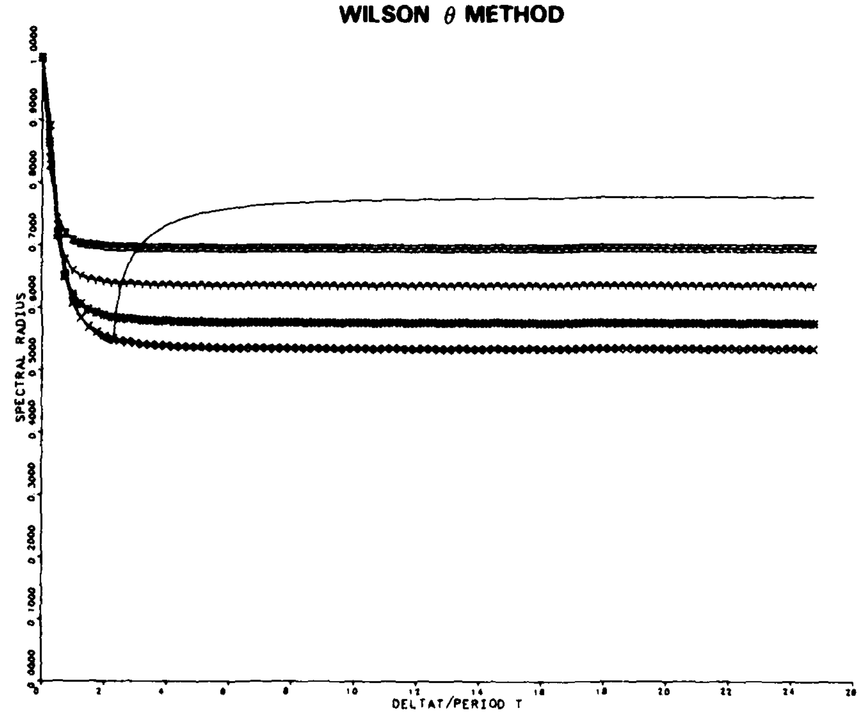
NEWMARK BETA METHOD β



β	γ
0.2500	0.5
0.3025	0.6
0.4225	0.8
0.6400	1.1
1.000	1.5

(A) BY NEWMARK β METHOD FOR $\xi = 0$

WILSON θ METHOD

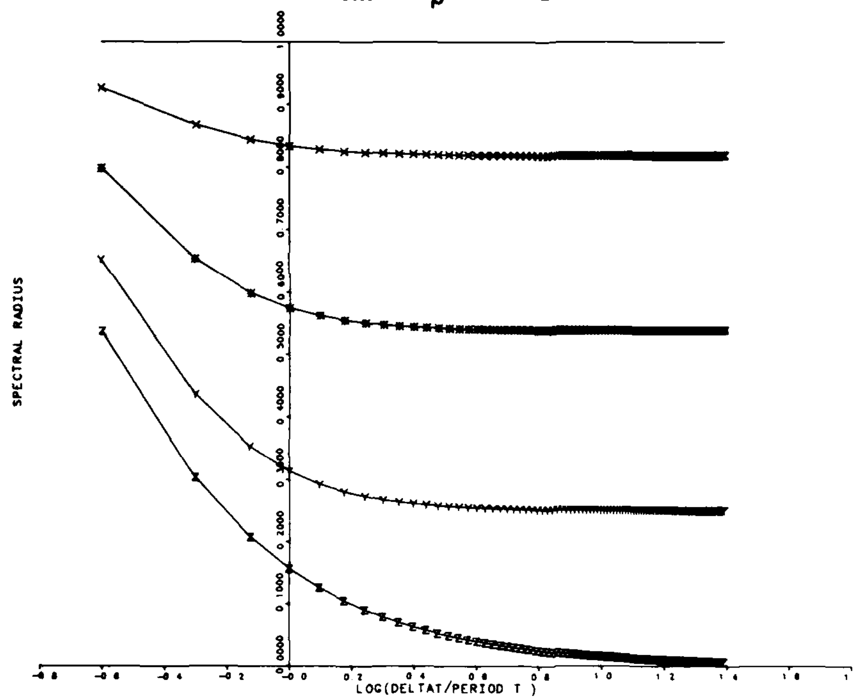


θ
1.4
2.4
2.0
1.7
1.5

(B) BY WILSON θ METHOD FOR $\xi = 0$

FIGURE 6-2. SPECTRAL RADIUS

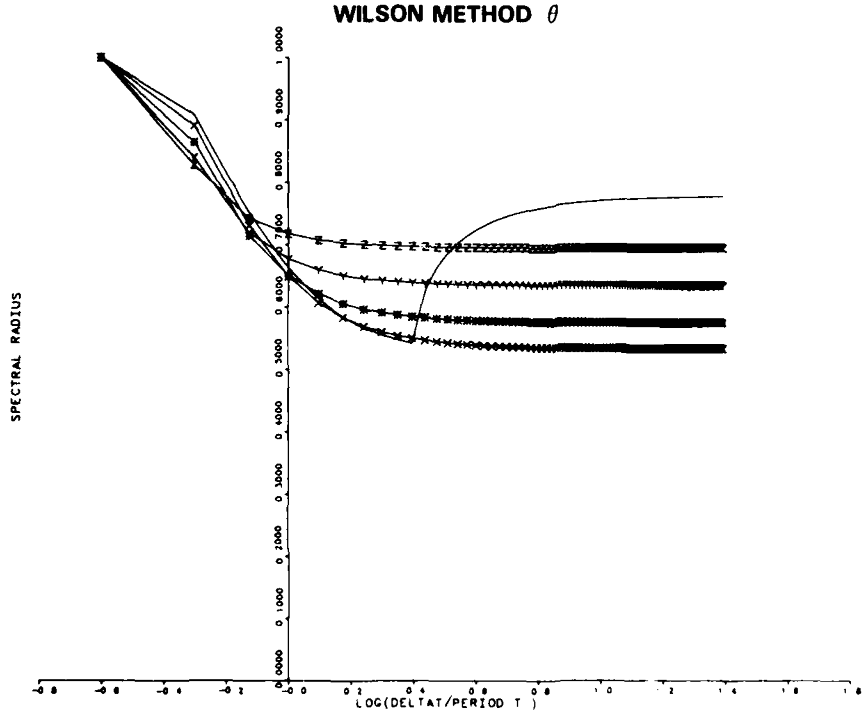
NEWMARK β METHOD



β	γ
0.2500	0.5
0.3025	0.6
0.4225	0.8
0.6400	1.1
1.0	1.5

(A) BY NEWMARK β METHOD FOR $\xi = 0$

WILSON METHOD θ

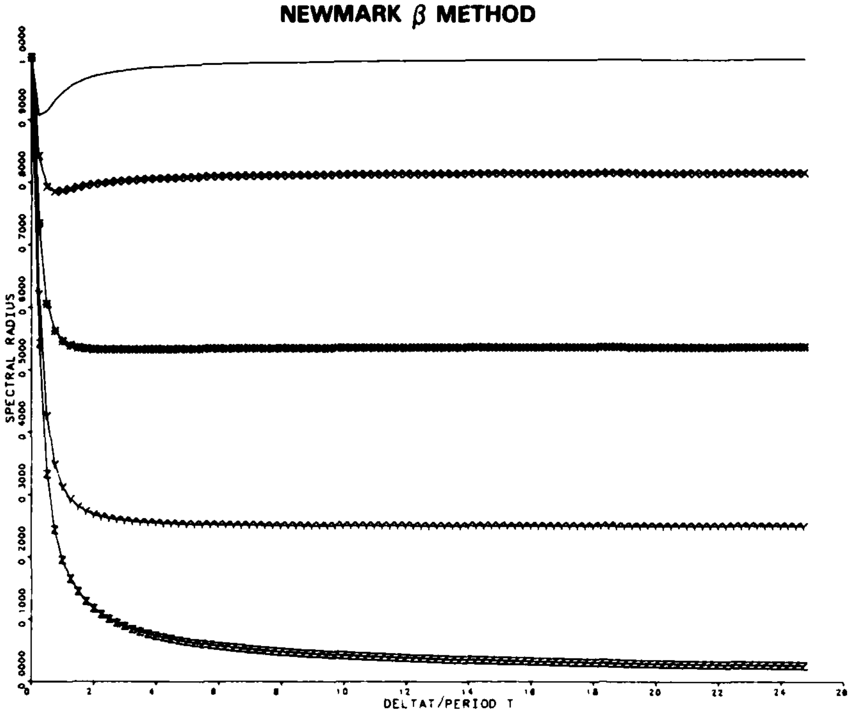


θ
1.4
2.4
2.0
1.7
1.5

(B) BY WILSON θ METHOD FOR $\xi = 0$

FIGURE 6-3. SPECTRAL RADIUS

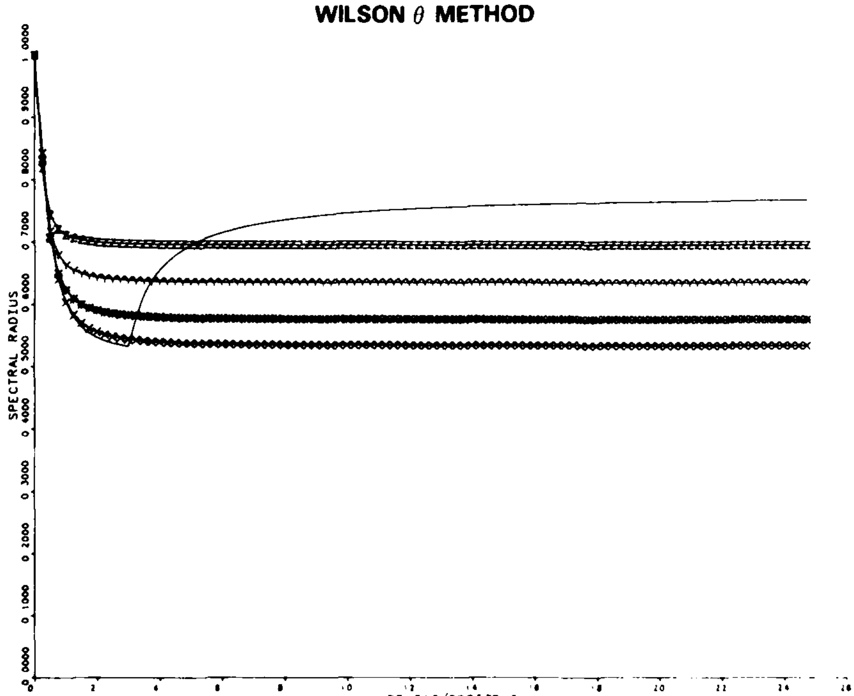
NEWMARK β METHOD



β	γ
0.25	0.5
0.3025	0.6
0.4225	0.8
0.64	1.1
1.0	1.5

(A) BY NEWMARK β METHOD FOR $\xi = 0.10$

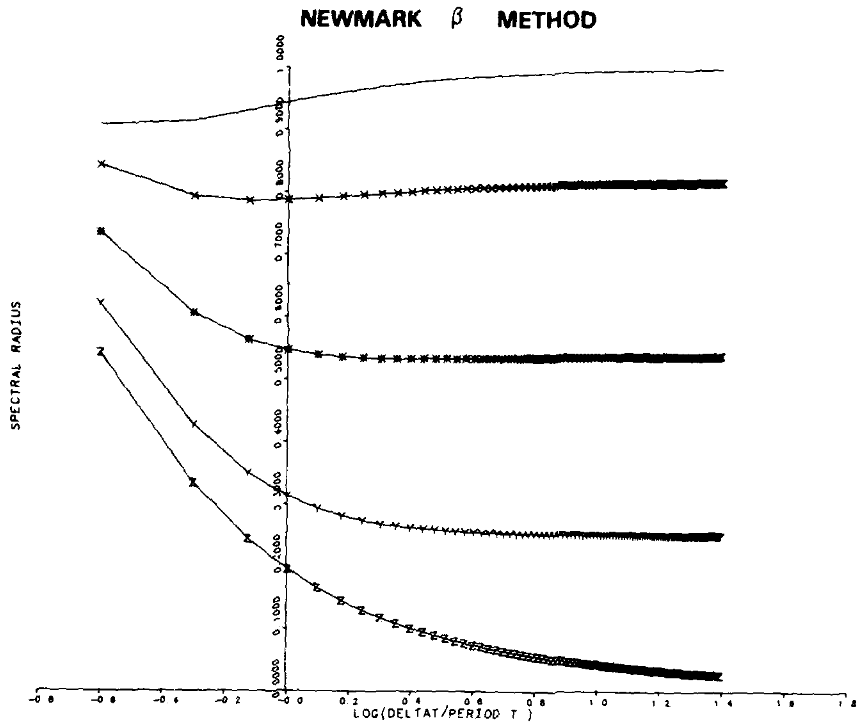
WILSON θ METHOD



θ
1.4
2.4
2.0
1.7
1.5

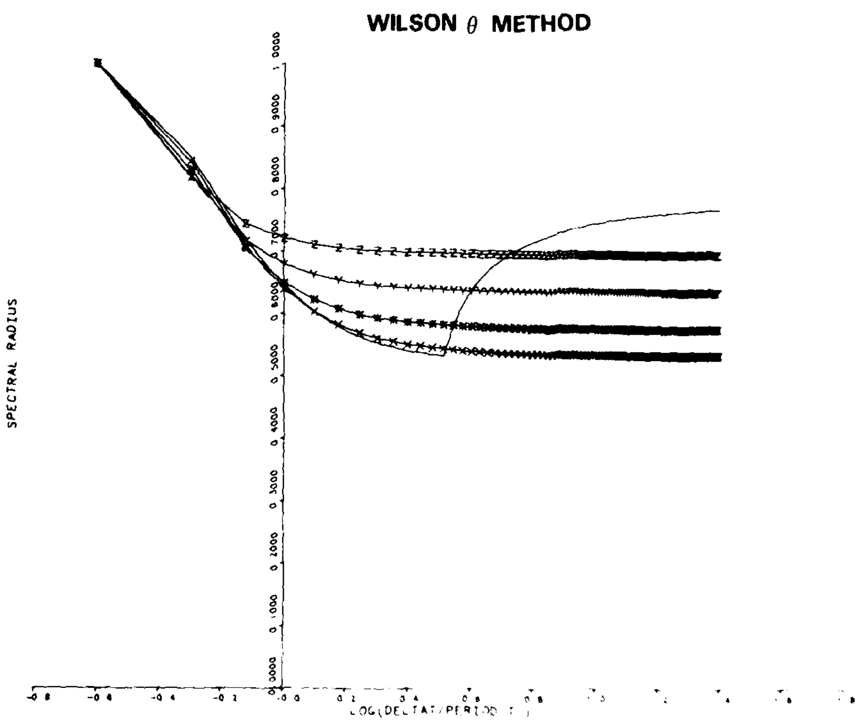
(B) BY WILSON θ METHOD FOR $\xi = 0.10$

FIGURE 6-4. SPECTRAL RADIUS



β	γ
0.25	0.5
0.3025	0.6
0.4225	0.8
0.64	1.1
1.0	1.5

(A) BY NEWMARK β METHOD FOR $\xi = 0.10$

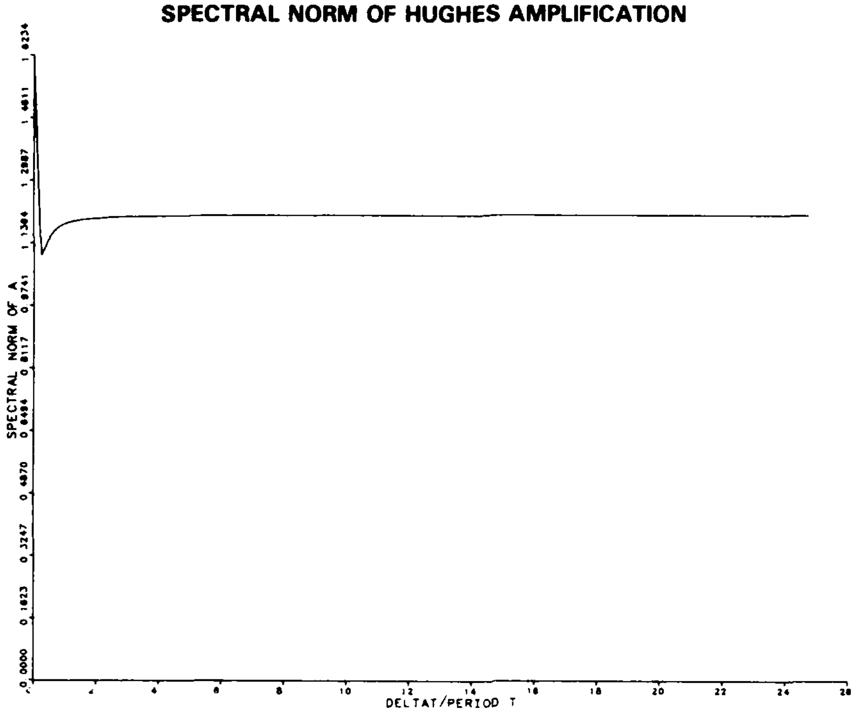


θ
1.4
2.4
2.0
1.7
1.5

(B) BY WILSON θ METHOD FOR $\xi = 0.10$

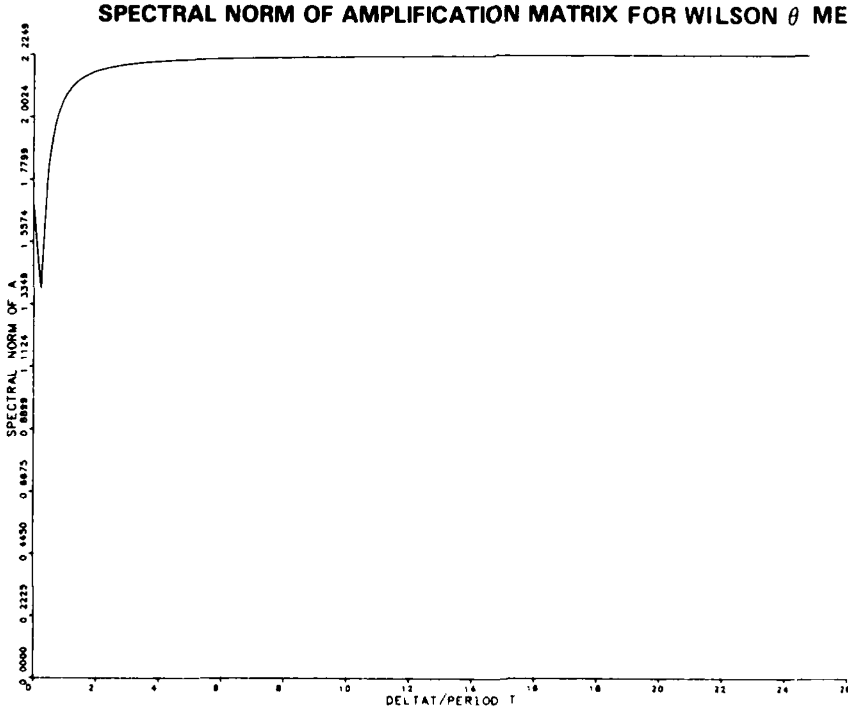
FIGURE 6-5. SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



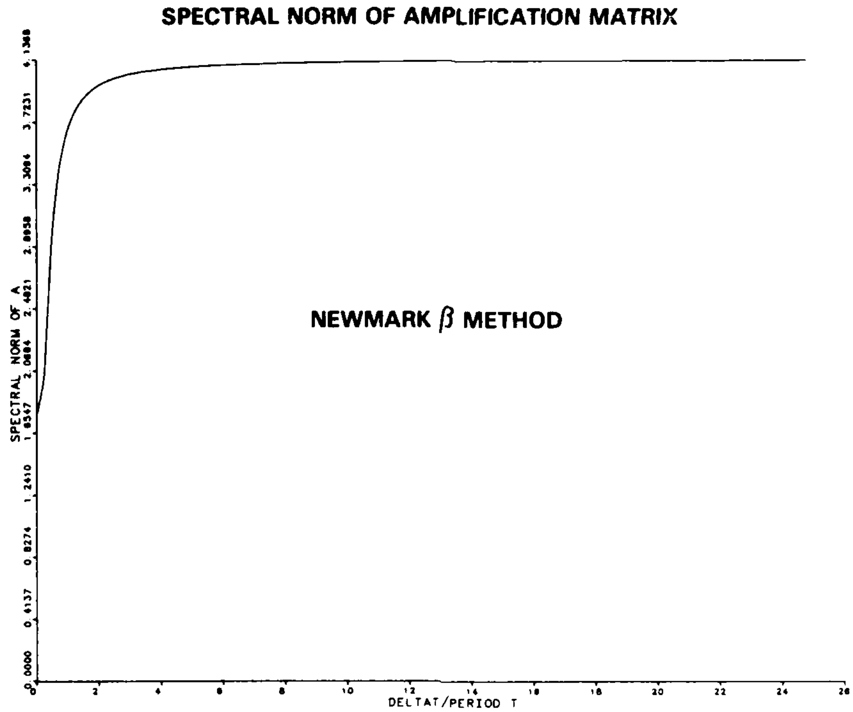
(A) OF HUGHES AMPLIFICATION MATRIX $\underset{\sim}{A}$ FOR θ = 1.4, ξ = 0.10, γ = 0.10, β = 0.916275

SPECTRAL NORM OF AMPLIFICATION MATRIX FOR WILSON θ METHOD

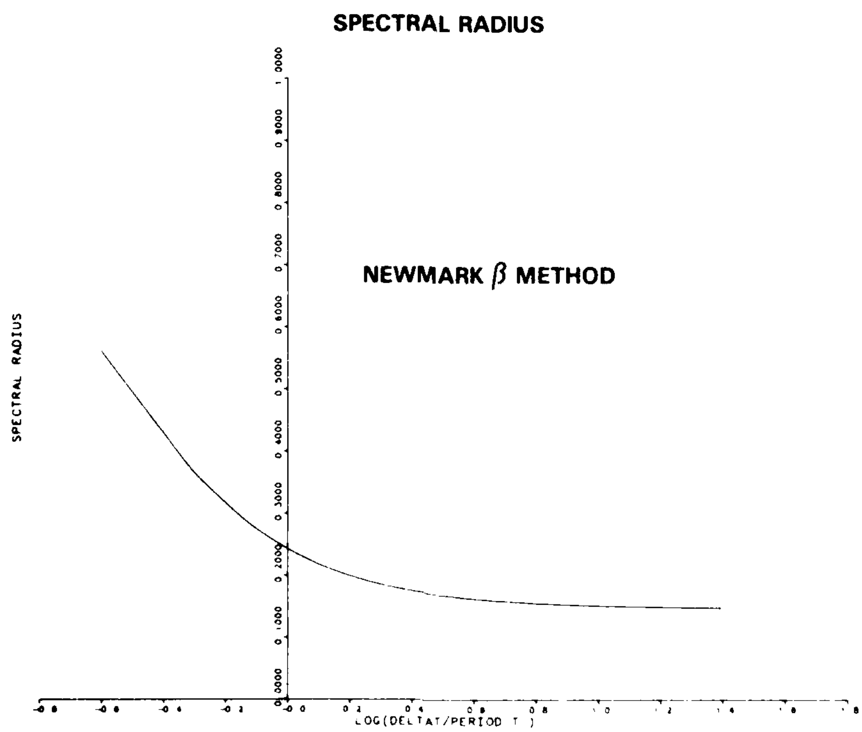


(B) OF WILSON θ AMPLIFICATION MATRIX $\underset{\sim}{A}$ FOR θ = 1.4, ξ = 0.10

FIGURE 6-6. SPECTRAL NORM



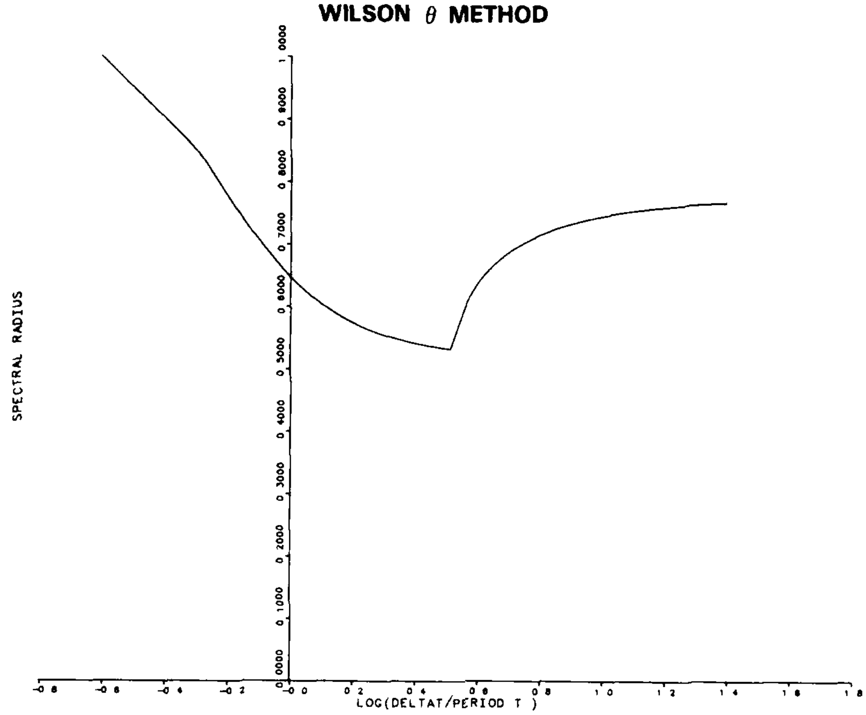
(A) OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY NEWMARK β METHOD FOR β = 0.916275, γ = 1.396296, ξ = 0.10



(B) BY NEWMARK β METHOD FOR β = 0.916275, γ = 1.396296, ξ = 0.10

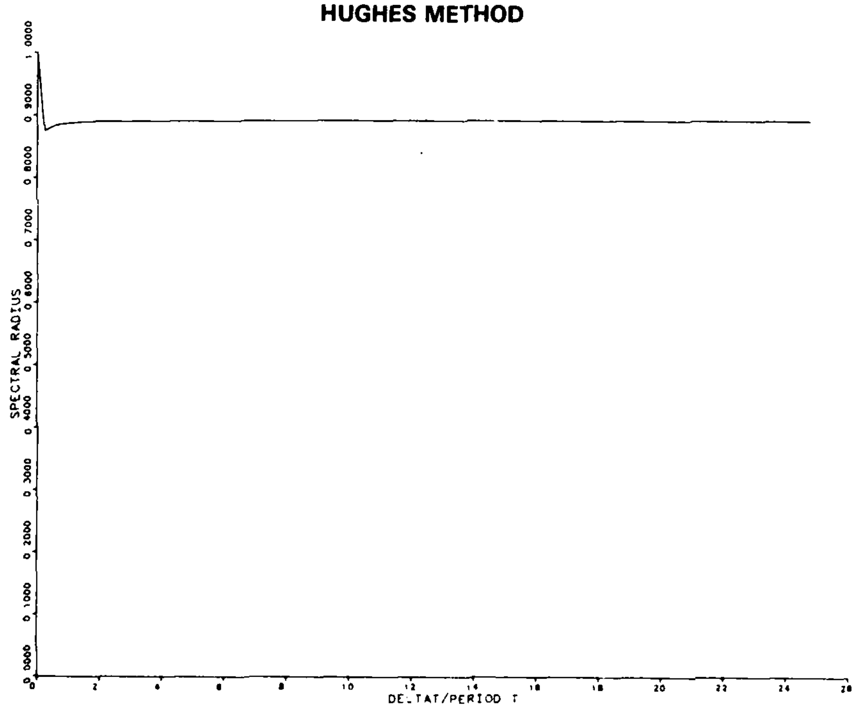
FIGURE 6-7. SPECTRAL NORM AND RADIUS

WILSON θ METHOD



(A) BY WILSON θ METHOD FOR θ = 1.4, ξ = 0.10

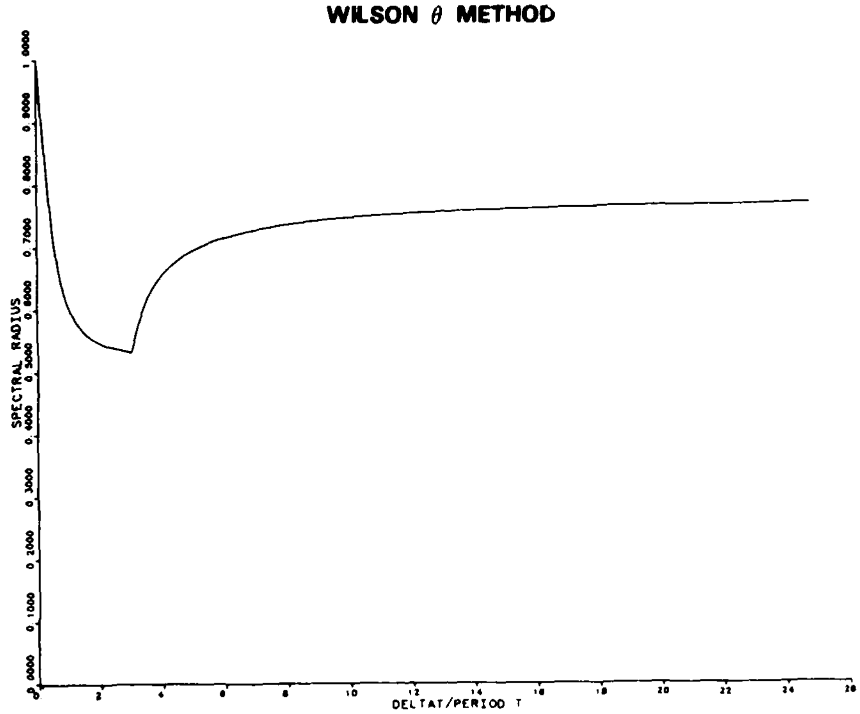
HUGHES METHOD



(B) BY HUGHES METHOD FOR θ = 1.4, ξ = 0.10, α = 0.10, β = 0.916275, γ = 1.396296

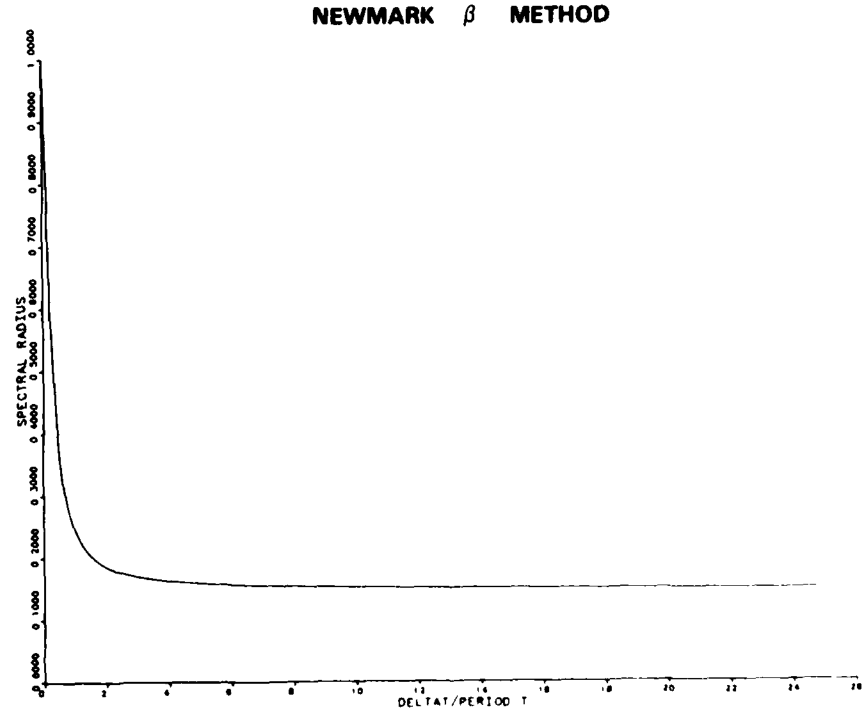
FIGURE 6-8. SPECTRAL RADIUS

WILSON θ METHOD



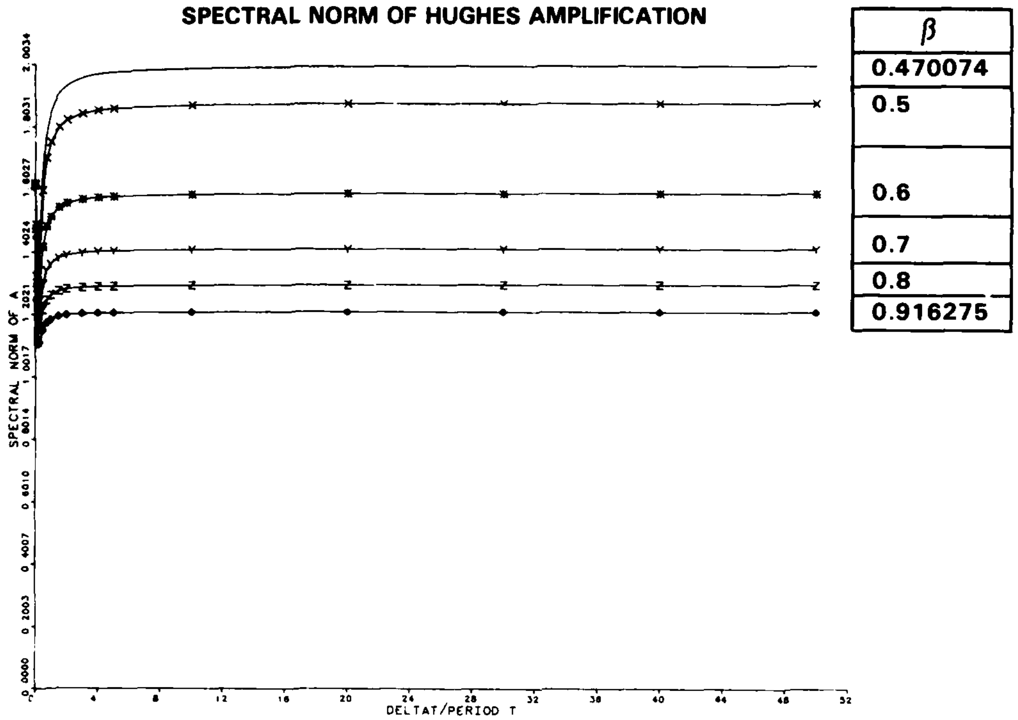
(A) BY WILSON θ METHOD FOR θ = 1.4. ξ = 0.10

NEWMARK β METHOD

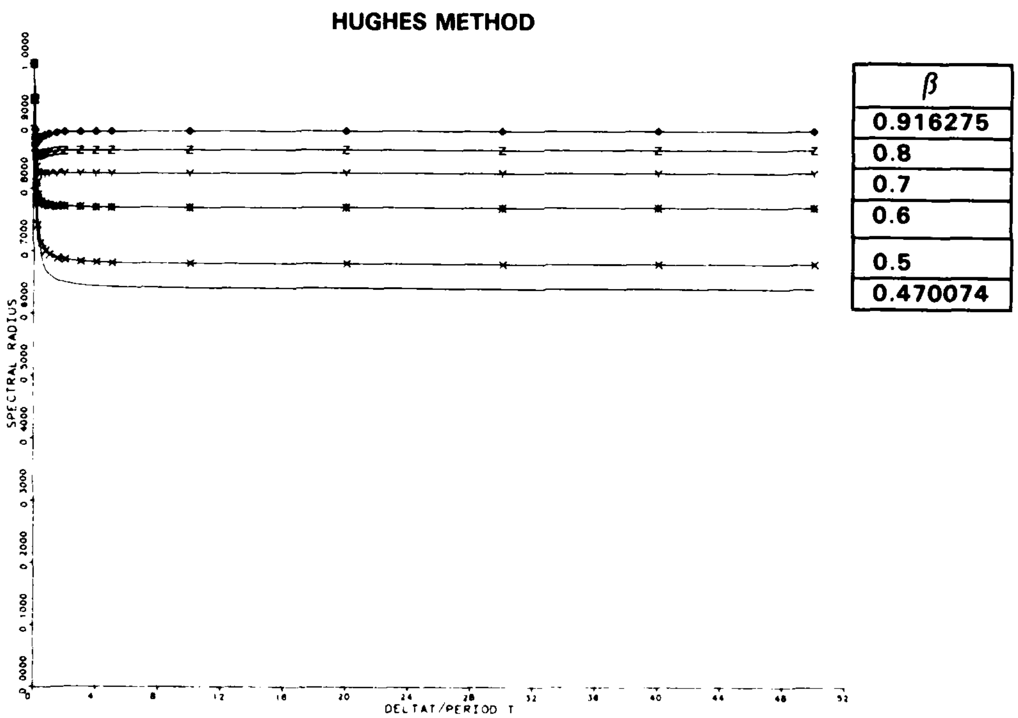


(B) BY NEWMARK β METHOD FOR β = 0.916275, γ = 1.396296, ξ = 0.10

FIGURE 6-9. SPECTRAL RADIUS

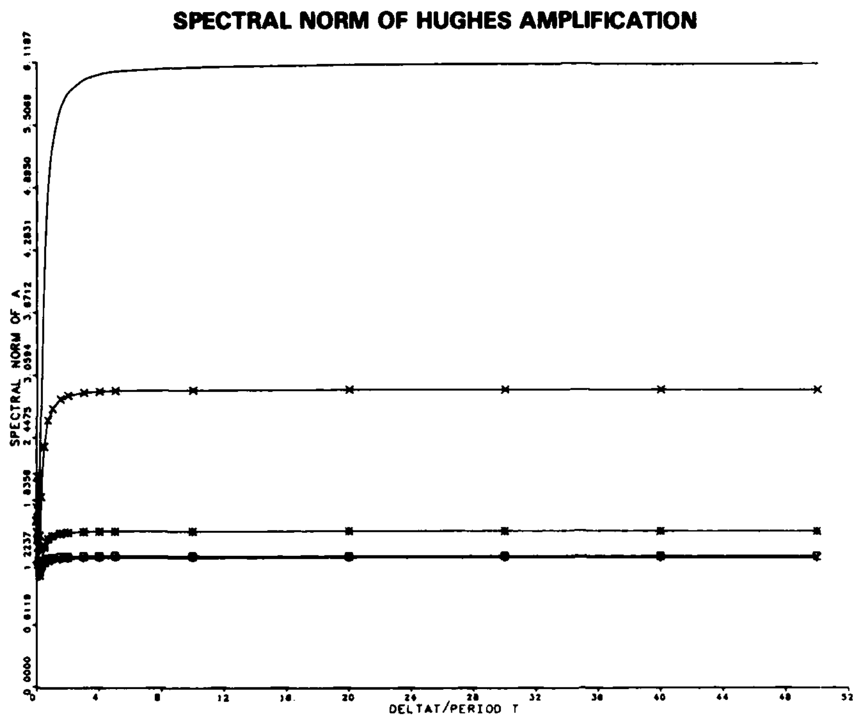


(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR α = −0.10, ξ = 0.10 γ = 1.396296, θ = 1.4



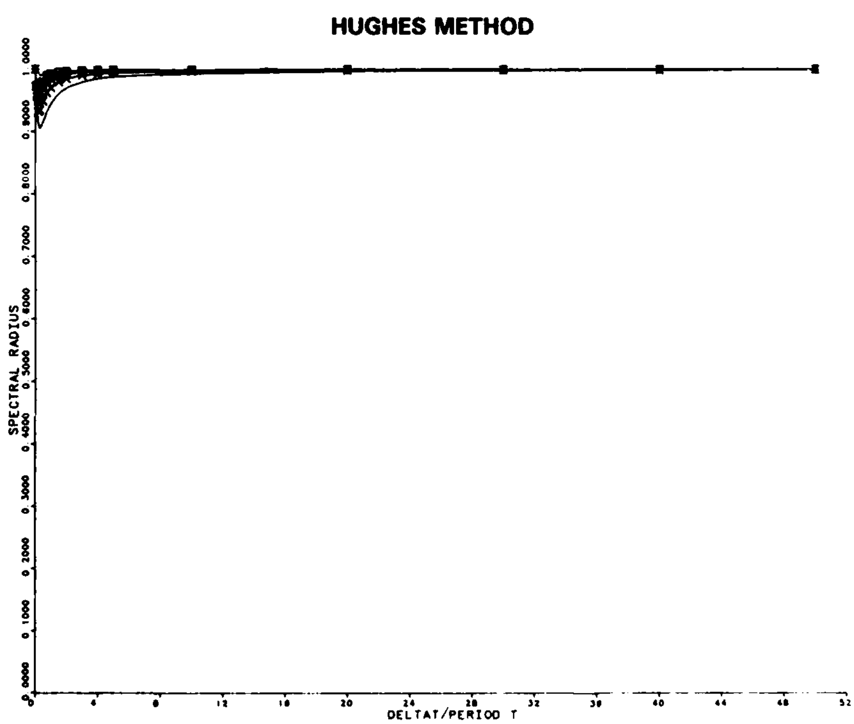
(B) SPECTRAL RADIUS BY HUGHES METHOD FOR α = −0.10, ξ = 0.10, γ = 1.396296, θ = 1.4

FIGURE 6-10. SPECTRAL NORM AND SPECTRAL RADIUS BY HUGHES METHOD



β	
0.25	Contin.
0.50	X
1.00	*
1.50	Y
2.00	Z

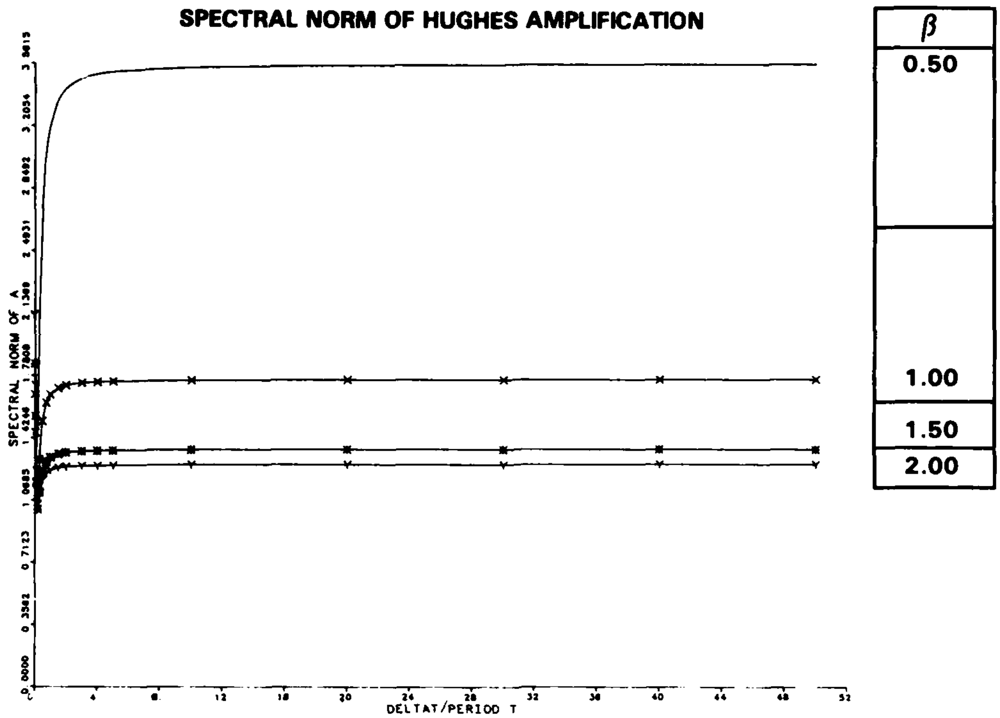
(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR $\alpha = 0.0, \gamma = 0.5, \theta = 1.0, \xi = 0.10$



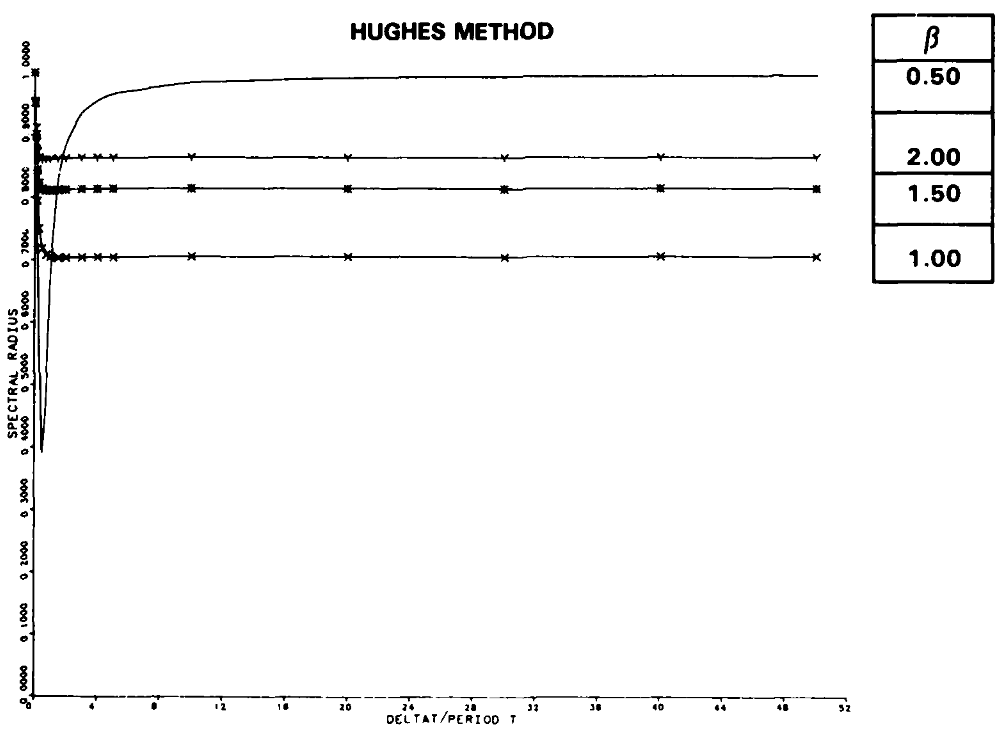
β	
0.25	Contin.
0.50	X
1.00	*
1.50	Y
2.00	Z

(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = 0.0, \gamma = 0.5, \theta = 1.0, \xi = 0.10$

FIGURE 6-11. SPECTRAL NORM AND SPECTRAL RADIUS

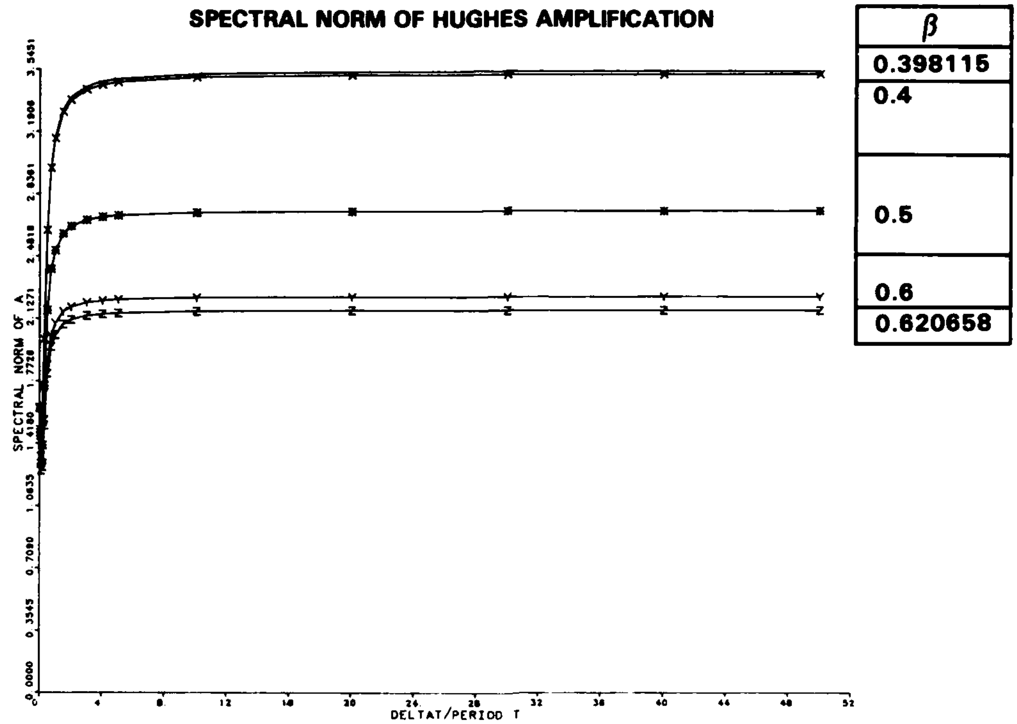


(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES FOR $\alpha = 0.0$, $\gamma = 1.0$, $\theta = 1.0$, $\xi = 0.10$

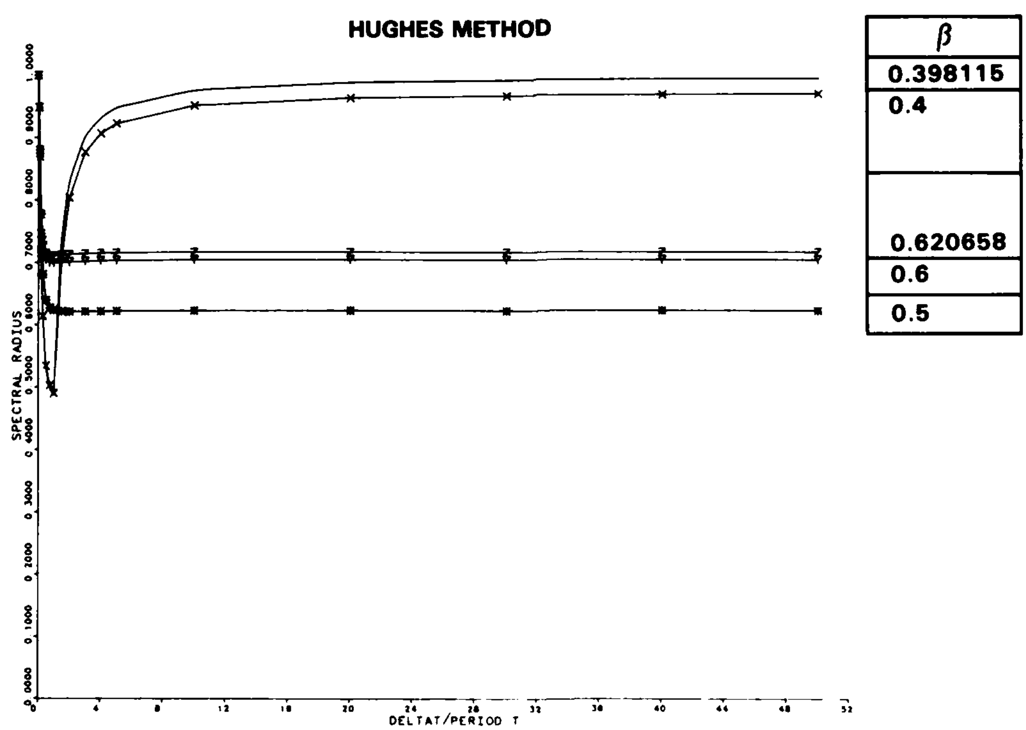


(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.0$, $\theta = 1.0$, $\xi = 0.10$

FIGURE 6-12. SPECTRAL NORM AND SPECTRAL RADIUS

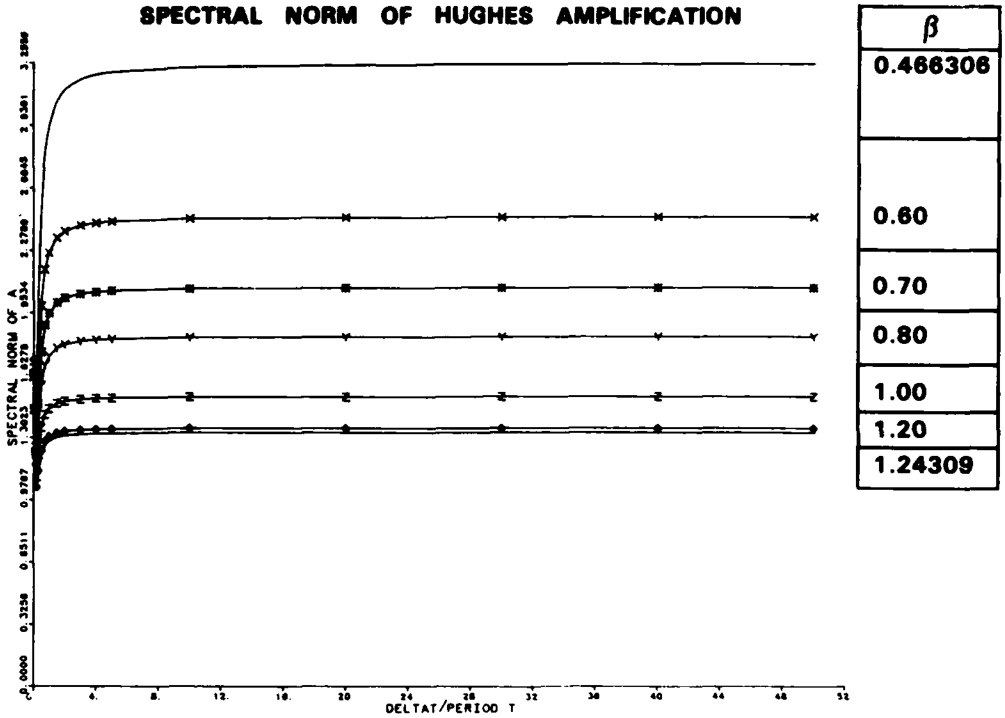


(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.01111$, $\theta = 1.1$, $\xi = 0.10$

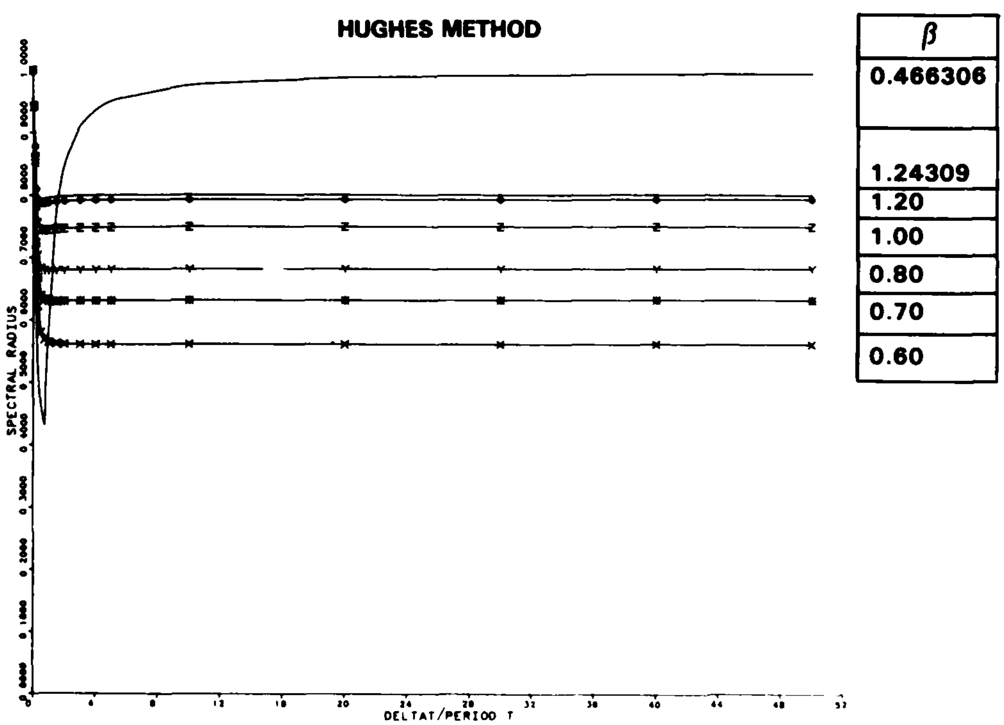


(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.01111$, $\theta = 1.1$, $\xi = 0.10$

FIGURE 6-13. SPECTRAL NORM AND SPECTRAL RADIUS

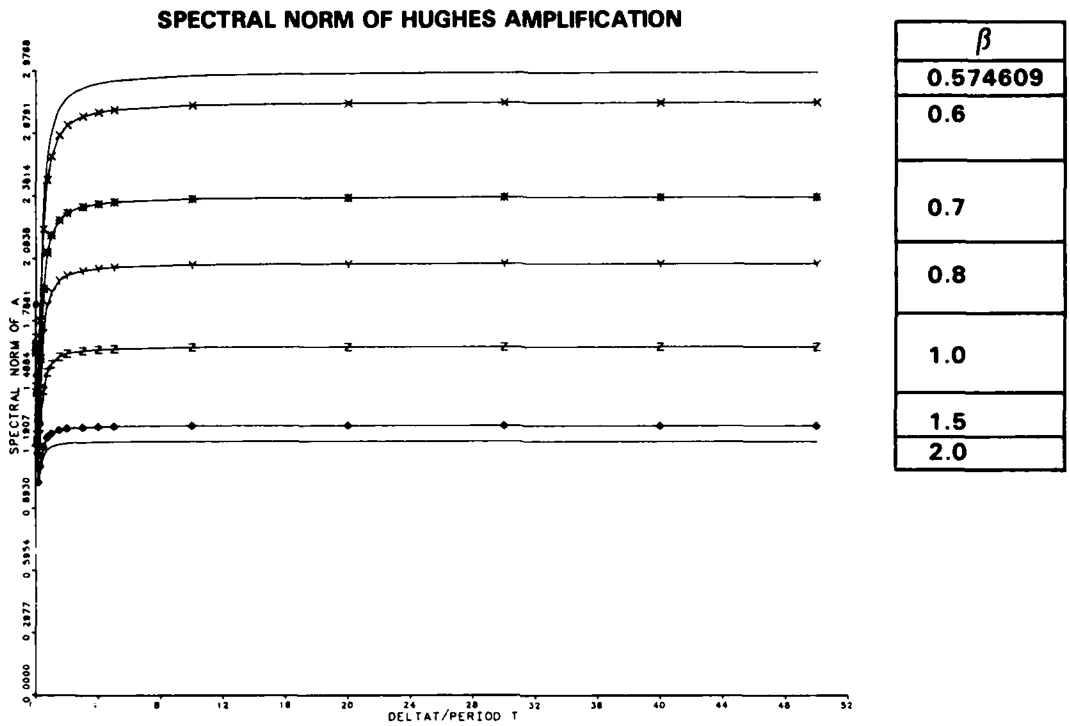


(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR $\alpha = 0.0, \gamma = 1.2, \theta = 1.1, \xi = 0.10$

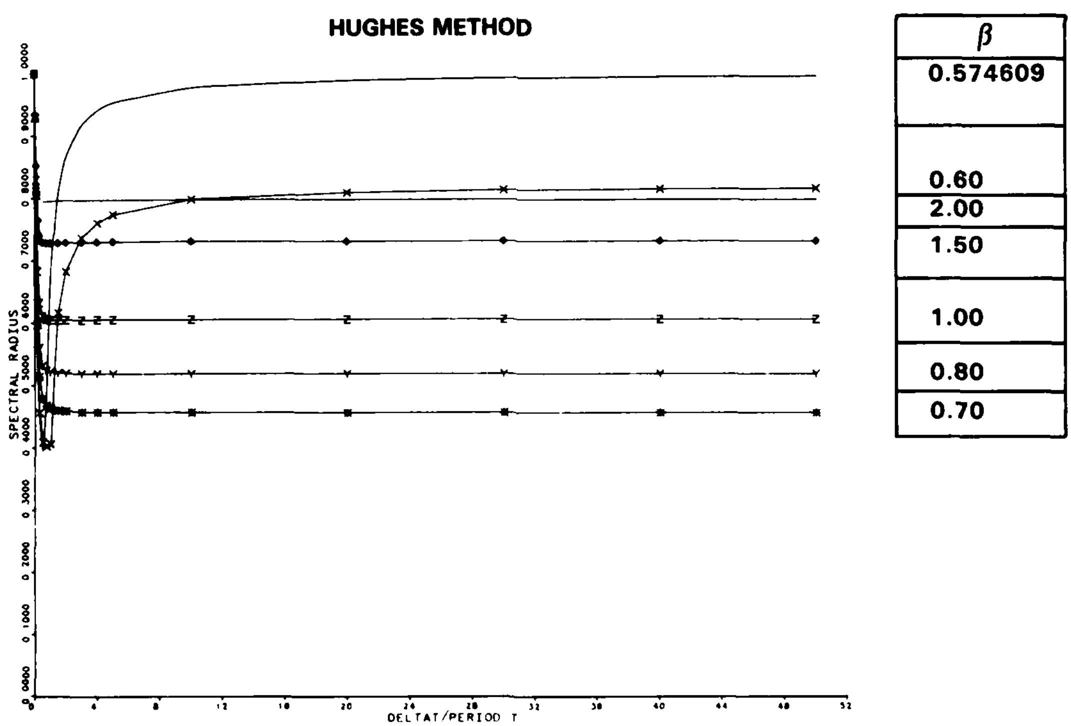


(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = 0.0, \gamma = 1.2, \theta = 1.1, \xi = 0.10$

FIGURE 6-14. SPECTRAL NORM AND SPECTRAL RADIUS



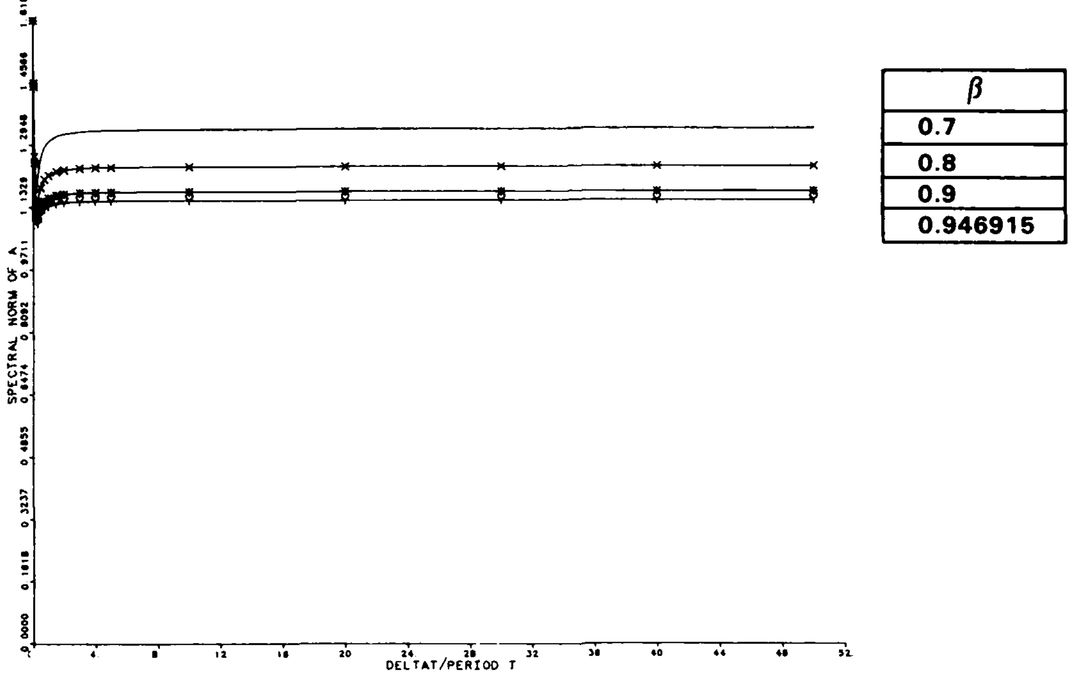
(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR α = 0.0, γ = 1.5, θ = 1.1, ξ = 0.10



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR α = 0.0, γ = 1.5, θ = 1.1, ξ = 0.10

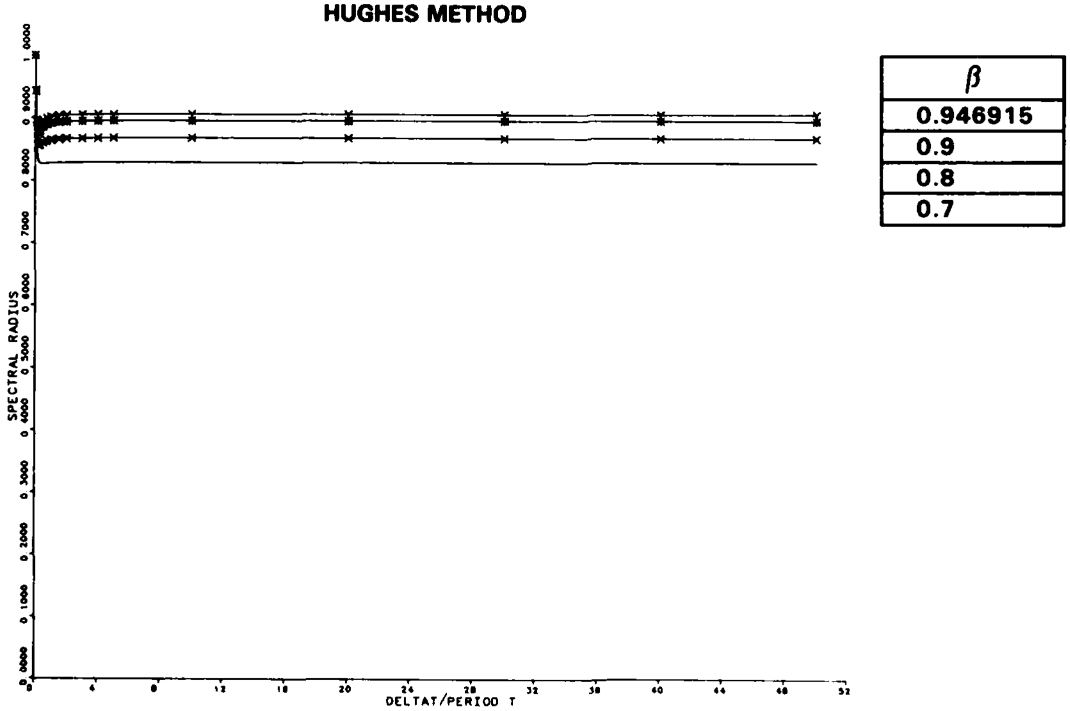
FIGURE 6-15. SPECTRAL NORM AND SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



(A) SPECTRAL NORM OF AMPLIFICATION MATRIX $\underset{\sim}{A}$ BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.3$, $\theta = 1.4$, $\xi = 0.1$

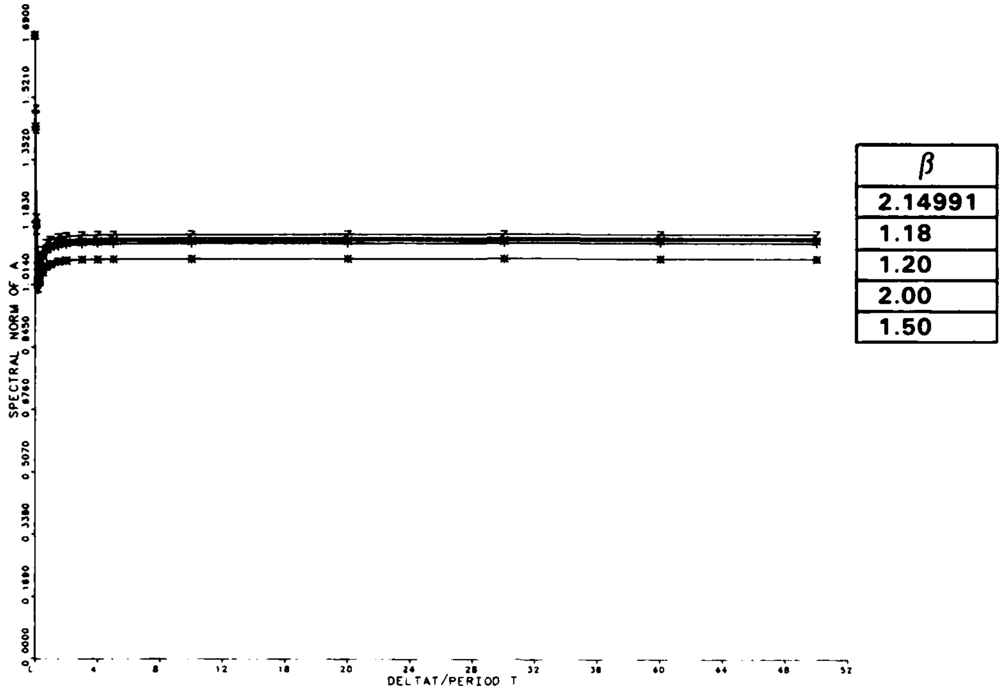
HUGHES METHOD



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.3$, $\theta = 1.4$, $\xi = 0.1$

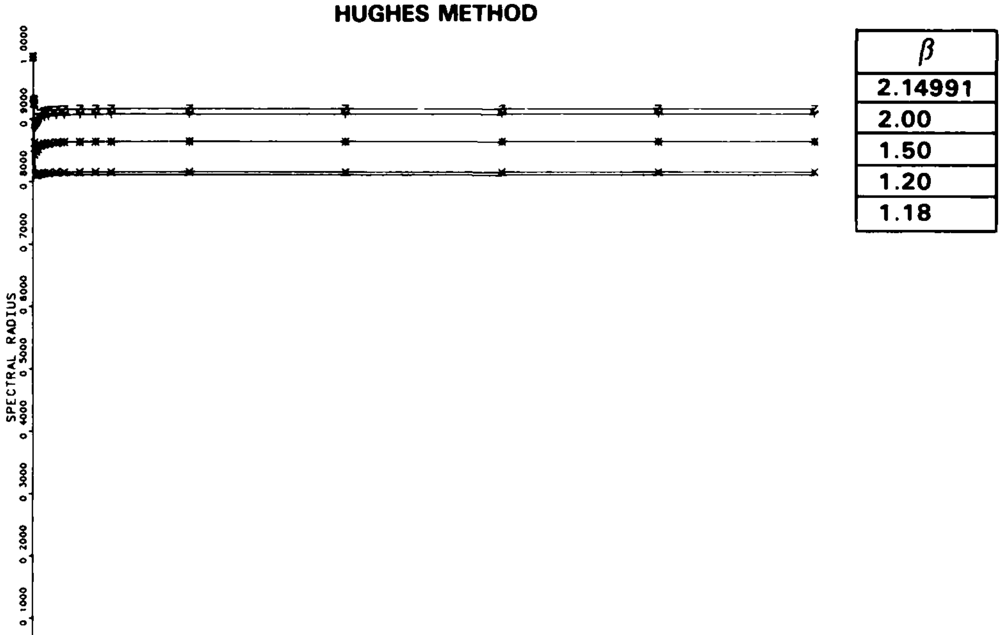
FIGURE 6-16. SPECTRAL NORM AND SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY HUGHES METHOD FOR α = 0.0, γ = 1.9, θ = 1.4, ξ = 0.1

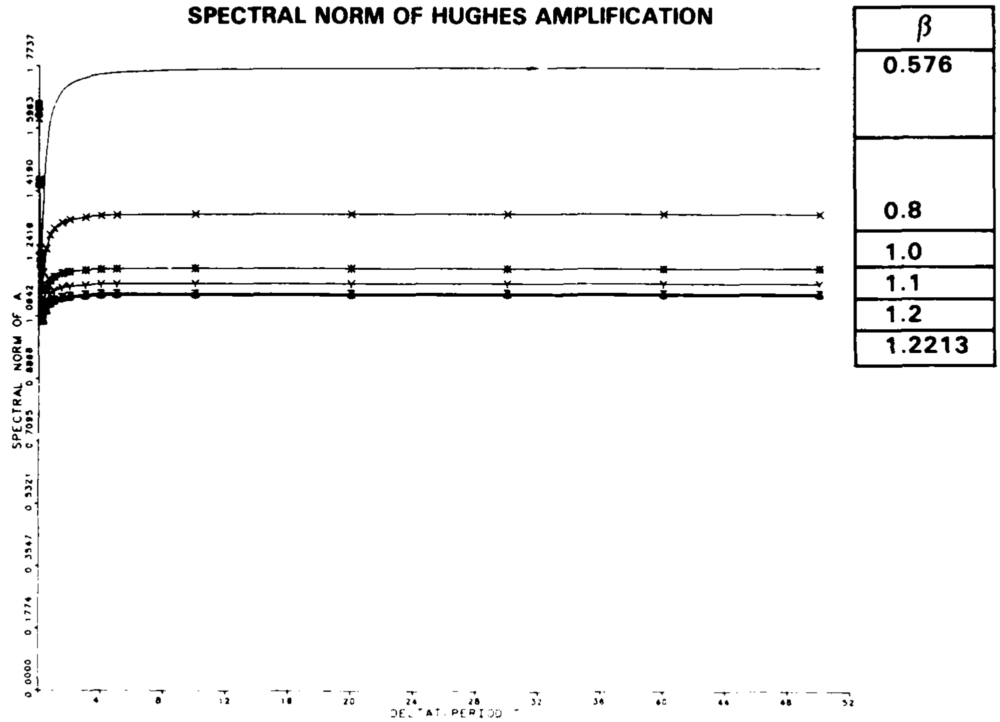
HUGHES METHOD



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR α = 0.0, γ = 1.9, θ = 1.4, ξ = 0.1

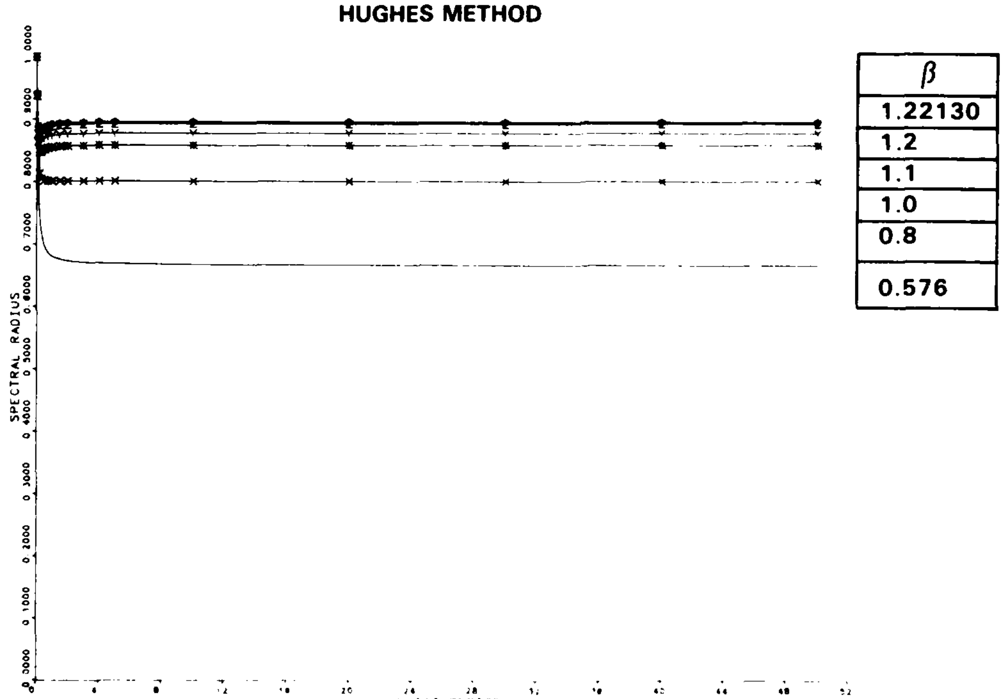
FIGURE 6-17. SPECTRAL NORM AND SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY HUGHES METHOD FOR $\alpha = -0.10, \gamma = 1.16, \theta = 1.4, \xi = 0.10$

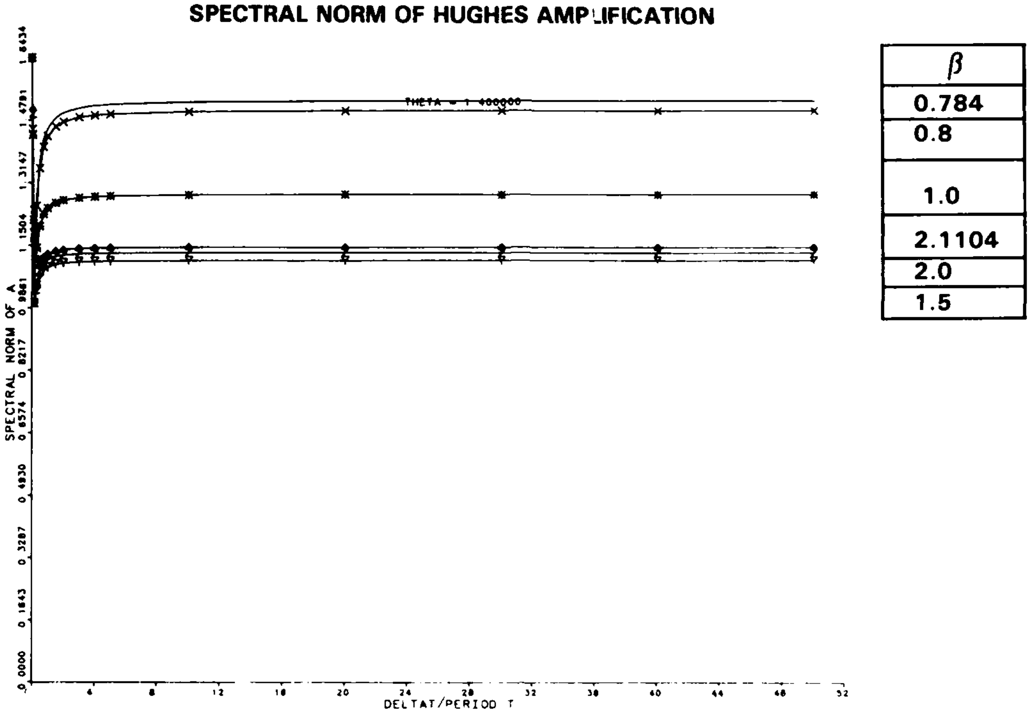
HUGHES METHOD



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = -0.10\ \gamma = 1.6,\ \theta = 1.4,\ \xi = 0.10$

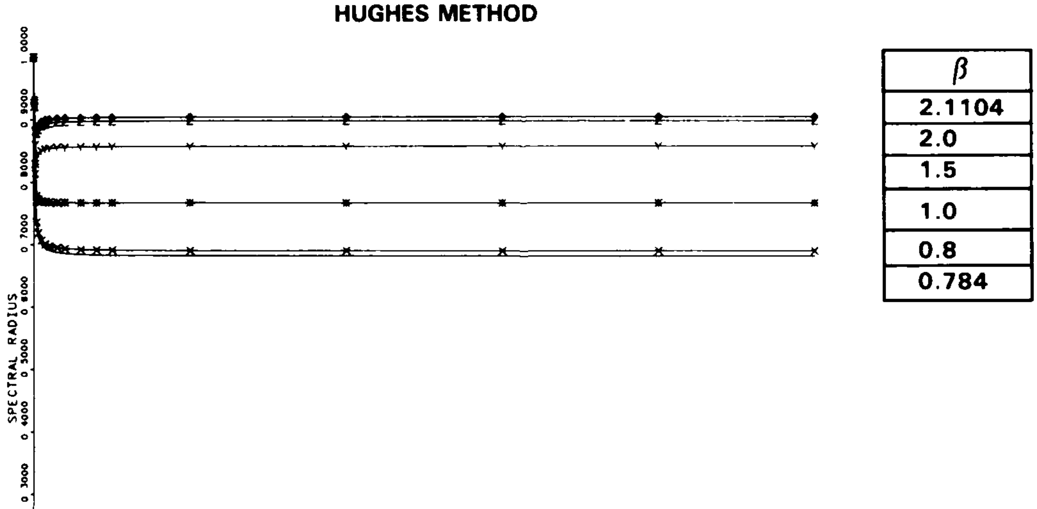
FIGURE 6-18. SPECTRAL NORM AND SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY . UGHES METHOD FOR $\alpha = -0.10$, $\gamma = 2.0$, $\theta \approx 1.4$, $\xi = 0.10$

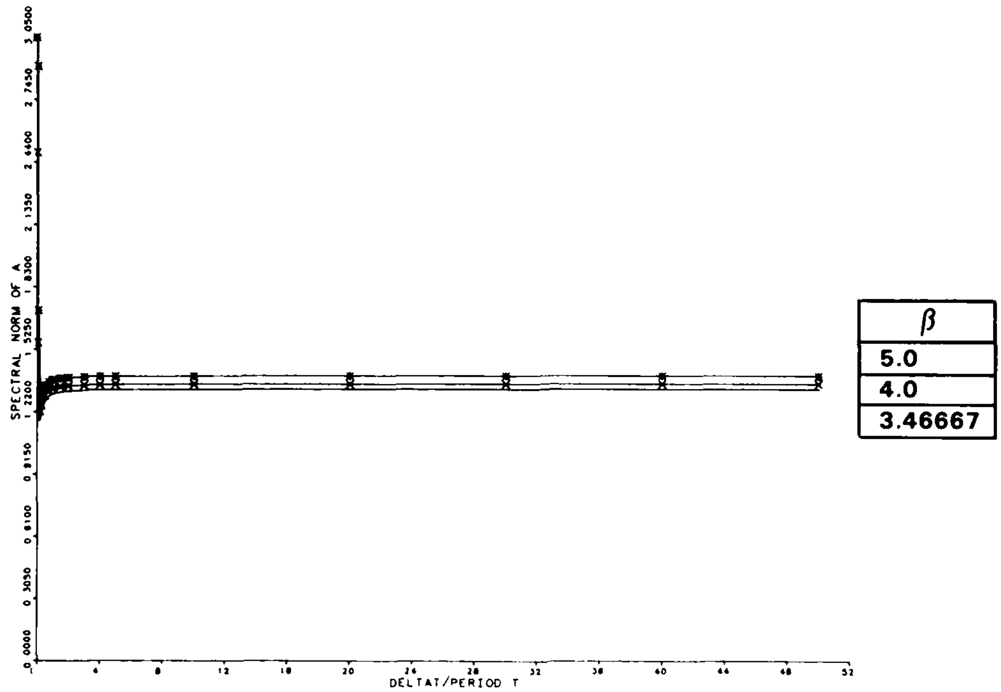
HUGHES METHOD



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = -0.10$, $\gamma = 2.0$, $\theta = 1.4$, $\xi = 0.10$

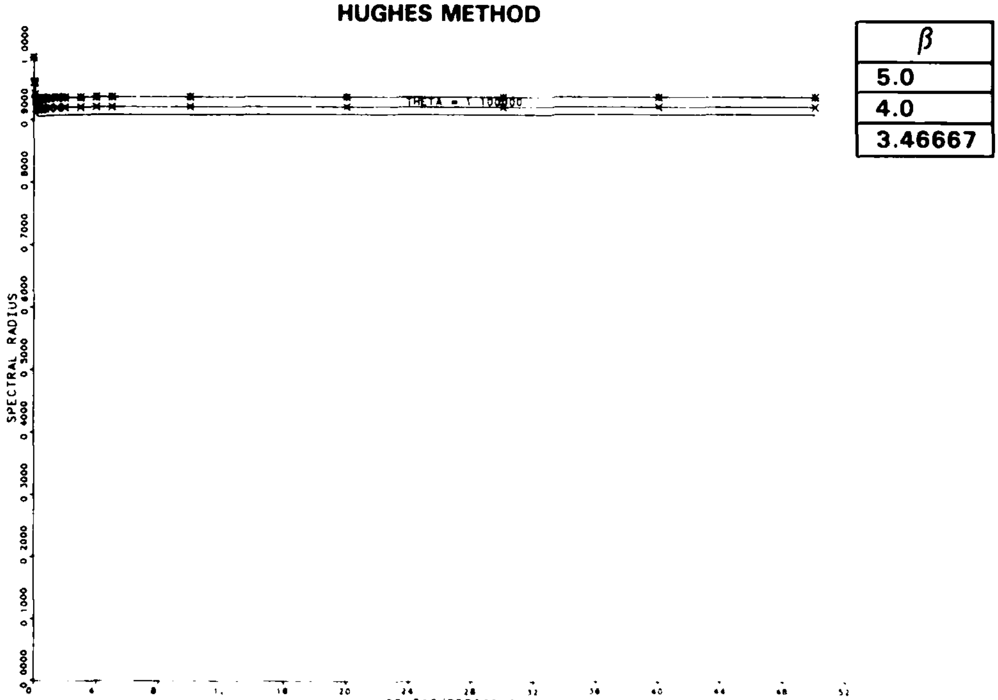
FIGURE 6-19. SPECTRAL NORM AND SPECTRAL RADIUS

SPECTRAL NORM OF HUGHES AMPLIFICATION



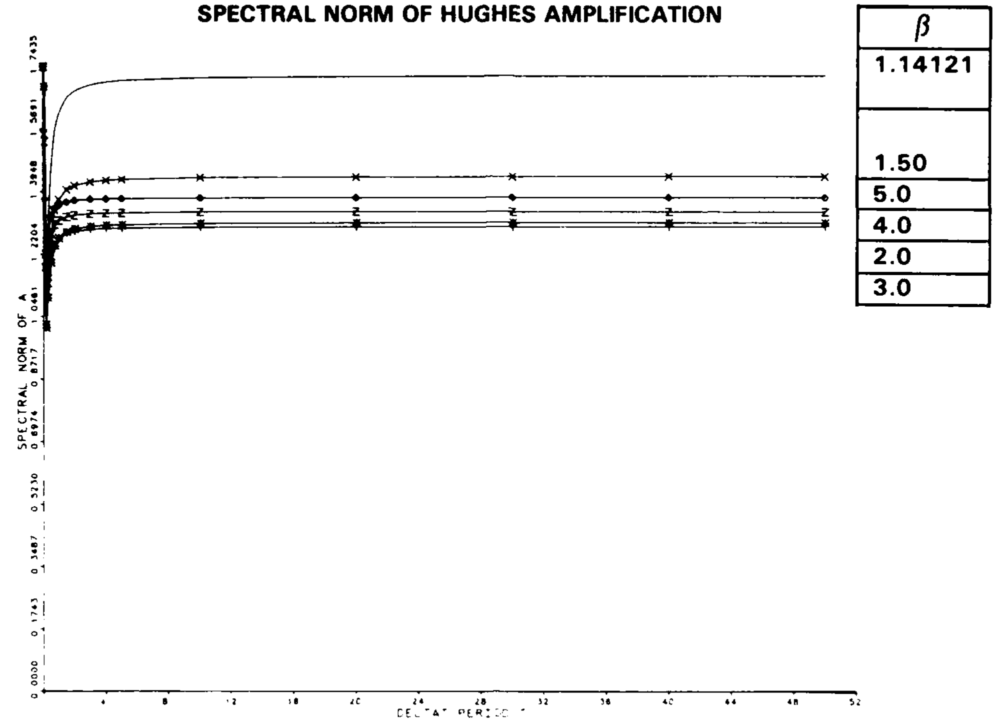
(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY HUGHES METHOD FOR $\alpha = -0.2$, $\gamma = 1.3$, $\theta = 1.1$, $\xi = 0.1$

HUGHES METHOD

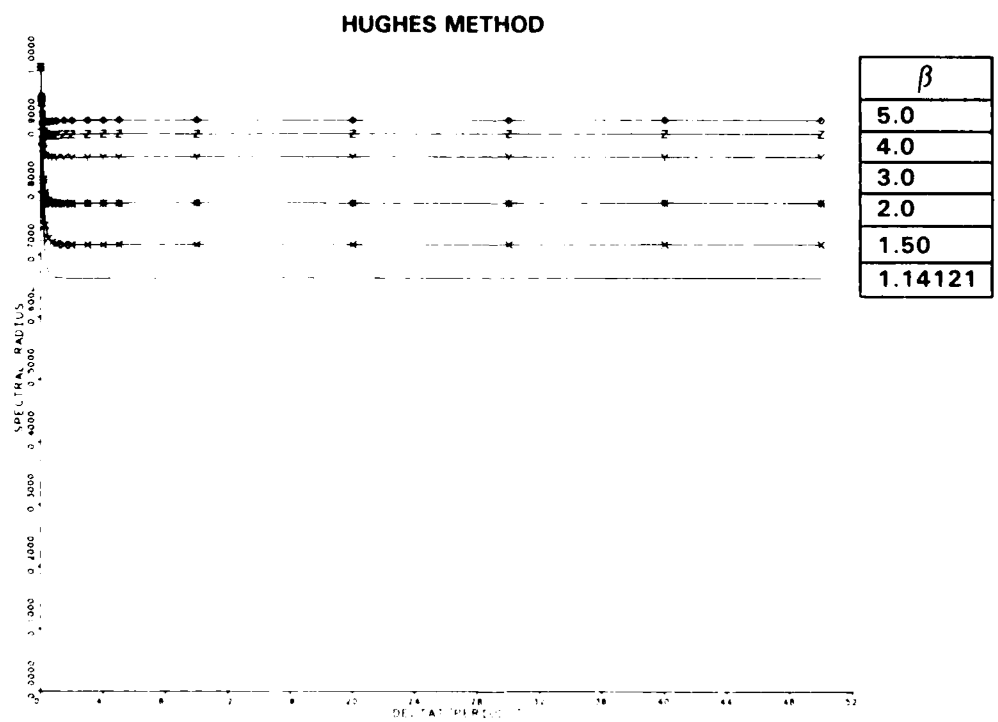


(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = -0.20$, $\gamma = 1.3$, $\theta = 1.1$, $\xi = 0.1$

FIGURE 6-20. SPECTRAL NORM AND SPECTRAL RADIUS

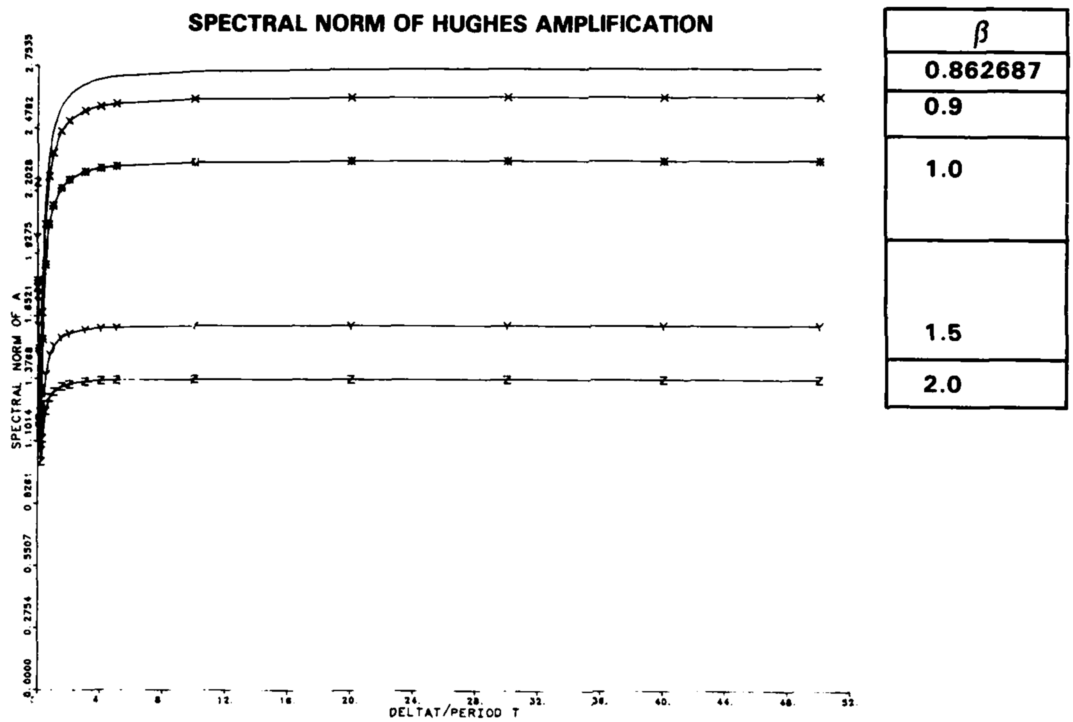


(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY HUGHES METHOD FOR $\alpha = -0.2, \gamma = 1.5, \theta = 1.1, \xi = 0.1$

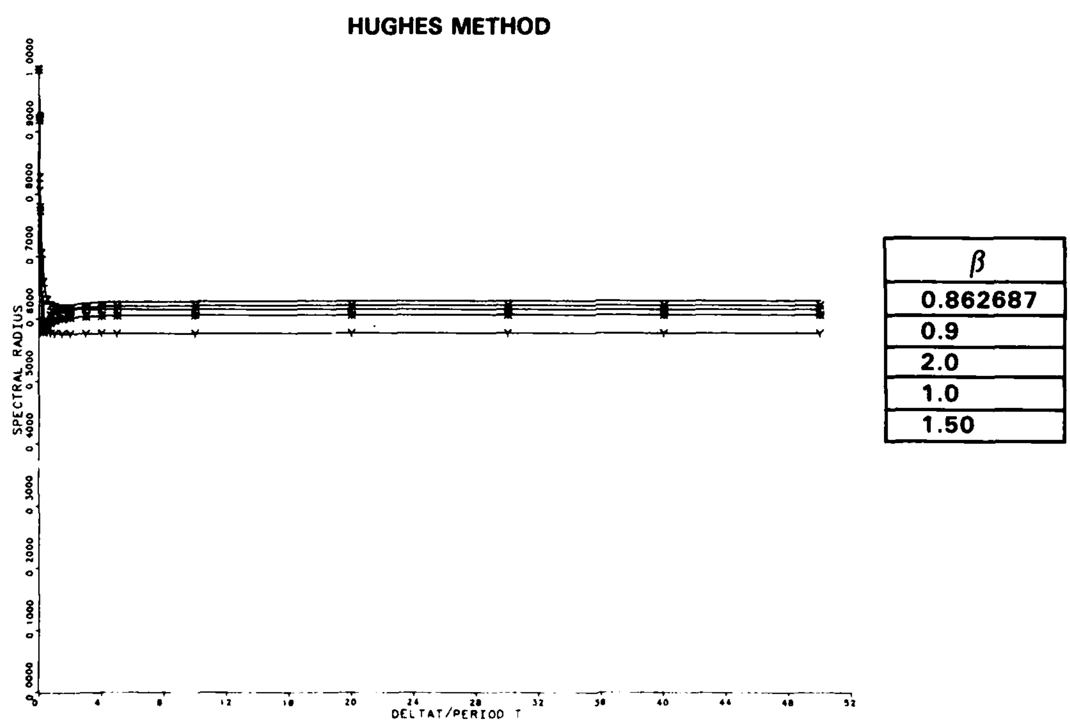


(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = -0.2, \gamma = 1.5, \theta = 1.1, \xi = 0.1$

FIGURE 6-21. SPECTRAL NORM AND SPECTRAL RADIUS



(A) SPECTRAL NORM OF AMPLIFICATION MATRIX BY HUGHES METHOD FOR $\alpha = -0.2, \gamma = 2.0, \theta = 1.1, \xi = 0.1$



(B) SPECTRAL RADIUS BY HUGHES METHOD FOR $\alpha = -0.2, \gamma = 2.0, \theta = 1.1, \xi = 0.1$

FIGURE 6-22. SPECTRAL NORM AND SPECTRAL RADIUS

TABLE 6-1. MATRIX RELATING FIGURES AND TABLES TO EACH OTHER

FIGURE NO.	TABLE NO.
6-1, 6-2, 6-3	6-2, 6-3, 6-4, 6-5, 6-6
6-10	6-7
6-11	6-8
6-12	6-9
6-13	6-10
6-14	6-11
6-15	6-12
6-16	6-13
6-17	6-14
6-18	6-15
6-19	6-16
6-20	6-17
6-21	6-18
6-22	6-19

TABLE 6-2. SPECTRAL RADIUS AND NORMS BY WILSON θ AND NEWMARK β METHODS FOR ξ = 0

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0.000000E+00	0.100000E+01	0.175775E+01	0.100000E+01	0.168889E+01
0.250000E+00	0.908084E+00	0.206821E+01	0.100000E+01	0.238791E+01
0.500000E+00	0.751169E+00	0.318306E+01	0.100000E+01	0.430435E+01
0.750000E+00	0.661404E+00	0.364548E+01	0.100000E+01	0.515114E+01
0.100000E+01	0.612547E+00	0.384709E+01	0.100000E+01	0.553398E+01
0.125000E+01	0.584234E+00	0.394913E+01	0.100000E+01	0.573120E+01
0.150000E+01	0.566718E+00	0.400711E+01	0.100000E+01	0.584434E+01
0.175000E+01	0.555257E+00	0.404297E+01	0.100000E+01	0.591473E+01
0.200000E+01	0.547403E+00	0.406663E+01	0.100000E+01	0.596132E+01
0.225000E+01	0.541810E+00	0.408302E+01	0.100000E+01	0.599369E+01
0.250000E+01	0.630159E+00	0.409484E+01	0.100000E+01	0.601706E+01
0.275000E+01	0.665948E+00	0.410363E+01	0.100000E+01	0.603447E+01
0.300000E+01	0.688239E+00	0.411034E+01	0.100000E+01	0.604777E+01
0.325000E+01	0.703876E+00	0.411558E+01	0.100000E+01	0.605817E+01
0.350000E+01	0.715503E+00	0.411975E+01	0.100000E+01	0.606645E+01
0.375000E+01	0.724472E+00	0.412312E+01	0.100000E+01	0.607314E+01
0.400000E+01	0.731578E+00	0.412588E+01	0.100000E+01	0.607862E+01
0.425000E+01	0.737323E+00	0.412817E+01	0.100000E+01	0.608318E+01
0.450000E+01	0.742045E+00	0.413010E+01	0.100000E+01	0.608700E+01
0.475000E+01	0.745981E+00	0.413172E+01	0.100000E+01	0.609024E+01
0.500000E+01	0.749299E+00	0.413312E+01	0.100000E+01	0.609301E+01
0.525000E+01	0.752124E+00	0.413432E+01	0.100000E+01	0.609539E+01
0.550000E+01	0.754552E+00	0.413536E+01	0.100000E+01	0.609746E+01
0.575000E+01	0.756654E+00	0.413626E+01	0.100000E+01	0.609927E+01
0.600000E+01	0.758488E+00	0.413706E+01	0.100000E+01	0.610085E+01
0.625000E+01	0.760097E+00	0.413776E+01	0.100000E+01	0.610225E+01
0.650000E+01	0.761517E+00	0.413839E+01	0.100000E+01	0.610349E+01
0.675000E+01	0.762777E+00	0.413894E+01	0.100000E+01	0.610460E+01
0.700000E+01	0.763900E+00	0.413944E+01	0.100000E+01	0.610559E+01
0.725000E+01	0.764906E+00	0.413989E+01	0.100000E+01	0.610648E+01
0.750000E+01	0.765810E+00	0.414029E+01	0.100000E+01	0.610728E+01
0.775000E+01	0.766626E+00	0.414066E+01	0.100000E+01	0.610801E+01
0.800000E+01	0.767365E+00	0.414099E+01	0.100000E+01	0.610867E+01
0.825000E+01	0.768037E+00	0.414129E+01	0.100000E+01	0.610927E+01
0.850000E+01	0.768648E+00	0.414156E+01	0.100000E+01	0.610982E+01
0.875000E+01	0.769208E+00	0.414182E+01	0.100000E+01	0.611032E+01
0.900000E+01	0.769720E+00	0.414205E+01	0.100000E+01	0.611078E+01
0.925000E+01	0.770191E+00	0.414226E+01	0.100000E+01	0.611121E+01
0.950000E+01	0.770624E+00	0.414246E+01	0.100000E+01	0.611160E+01
0.975000E+01	0.771024E+00	0.414264E+01	0.100000E+01	0.611196E+01
0.100000E+02	0.771394E+00	0.414281E+01	0.100000E+01	0.611229E+01
0.102500E+02	0.771737E+00	0.414296E+01	0.100000E+01	0.611260E+01
0.105000E+02	0.772056E+00	0.414311E+01	0.100000E+01	0.611289E+01
0.107500E+02	0.772352E+00	0.414324E+01	0.100000E+01	0.611316E+01
0.110000E+02	0.772628E+00	0.414337E+01	0.100000E+01	0.611341E+01
0.112500E+02	0.772885E+00	0.414349E+01	0.100000E+01	0.611365E+01
0.115000E+02	0.773126E+00	0.414360E+01	0.100000E+01	0.611387E+01
0.117500E+02	0.773352E+00	0.414370E+01	0.100000E+01	0.611407E+01
0.120000E+02	0.773563E+00	0.414380E+01	0.100000E+01	0.611426E+01
0.122500E+02	0.773761E+00	0.414389E+01	0.100000E+01	0.611445E+01
0.125000E+02	0.773948E+00	0.414397E+01	0.100000E+01	0.611462E+01
0.127500E+02	0.774124E+00	0.414405E+01	0.100000E+01	0.611478E+01

NOTE CONSTANT VALUES ARE:

$\beta = 0.250 \qquad \theta = 1.4$

$\gamma = 0.50 \qquad \xi = 0$

TABLE 6-2. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0. 130000E+02	0. 774289E+00	0. 414413E+01	0. 100000E+01	0. 611493E+01
0. 132500E+02	0. 774445E+00	0. 414420E+01	0. 100000E+01	0. 611507E+01
0. 135000E+02	0. 774593E+00	0. 414427E+01	0. 100000E+01	0. 611521E+01
0. 137500E+02	0. 774732E+00	0. 414433E+01	0. 100000E+01	0. 611533E+01
0. 140000E+02	0. 774864E+00	0. 414439E+01	0. 100000E+01	0. 611545E+01
0. 142500E+02	0. 774989E+00	0. 414445E+01	0. 100000E+01	0. 611557E+01
0. 145000E+02	0. 775108E+00	0. 414451E+01	0. 100000E+01	0. 611568E+01
0. 147500E+02	0. 775220E+00	0. 414456E+01	0. 100000E+01	0. 611578E+01
0. 150000E+02	0. 775327E+00	0. 414461E+01	0. 100000E+01	0. 611588E+01
0. 152500E+02	0. 775429E+00	0. 414465E+01	0. 100000E+01	0. 611597E+01
0. 155000E+02	0. 775526E+00	0. 414470E+01	0. 100000E+01	0. 611606E+01
0. 157500E+02	0. 775618E+00	0. 414474E+01	0. 100000E+01	0. 611615E+01
0. 160000E+02	0. 775706E+00	0. 414478E+01	0. 100000E+01	0. 611623E+01
0. 162500E+02	0. 775790E+00	0. 414482E+01	0. 100000E+01	0. 611630E+01
0. 165000E+02	0. 775870E+00	0. 414486E+01	0. 100000E+01	0. 611638E+01
0. 167500E+02	0. 775946E+00	0. 414489E+01	0. 100000E+01	0. 611645E+01
0. 170000E+02	0. 776020E+00	0. 414493E+01	0. 100000E+01	0. 611651E+01
0. 172500E+02	0. 776090E+00	0. 414496E+01	0. 100000E+01	0. 611658E+01
0. 175000E+02	0. 776156E+00	0. 414499E+01	0. 100000E+01	0. 611664E+01
0. 177500E+02	0. 776221E+00	0. 414502E+01	0. 100000E+01	0. 611670E+01
0. 180000E+02	0. 776282E+00	0. 414505E+01	0. 100000E+01	0. 611676E+01
0. 182500E+02	0. 776341E+00	0. 414508E+01	0. 100000E+01	0. 611681E+01
0. 185000E+02	0. 776398E+00	0. 414510E+01	0. 100000E+01	0. 611686E+01
0. 187500E+02	0. 776452E+00	0. 414513E+01	0. 100000E+01	0. 611691E+01
0. 190000E+02	0. 776504E+00	0. 414515E+01	0. 100000E+01	0. 611696E+01
0. 192500E+02	0. 776554E+00	0. 414517E+01	0. 100000E+01	0. 611701E+01
0. 195000E+02	0. 776602E+00	0. 414520E+01	0. 100000E+01	0. 611705E+01
0. 197500E+02	0. 776649E+00	0. 414522E+01	0. 100000E+01	0. 611709E+01
0. 200000E+02	0. 776694E+00	0. 414524E+01	0. 100000E+01	0. 611713E+01
0. 202500E+02	0. 776737E+00	0. 414526E+01	0. 100000E+01	0. 611717E+01
0. 205000E+02	0. 776778E+00	0. 414528E+01	0. 100000E+01	0. 611721E+01
0. 207500E+02	0. 776818E+00	0. 414530E+01	0. 100000E+01	0. 611725E+01
0. 210000E+02	0. 776856E+00	0. 414531E+01	0. 100000E+01	0. 611728E+01
0. 212500E+02	0. 776894E+00	0. 414533E+01	0. 100000E+01	0. 611732E+01
0. 215000E+02	0. 776930E+00	0. 414535E+01	0. 100000E+01	0. 611735E+01
0. 217500E+02	0. 776964E+00	0. 414536E+01	0. 100000E+01	0. 611738E+01
0. 220000E+02	0. 776998E+00	0. 414538E+01	0. 100000E+01	0. 611741E+01
0. 222500E+02	0. 777030E+00	0. 414539E+01	0. 100000E+01	0. 611744E+01
0. 225000E+02	0. 777061E+00	0. 414541E+01	0. 100000E+01	0. 611747E+01
0. 227500E+02	0. 777091E+00	0. 414542E+01	0. 100000E+01	0. 611750E+01
0. 230000E+02	0. 777121E+00	0. 414544E+01	0. 100000E+01	0. 611753E+01
0. 232500E+02	0. 777149E+00	0. 414545E+01	0. 100000E+01	0. 611755E+01
0. 235000E+02	0. 777176E+00	0. 414546E+01	0. 100000E+01	0. 611758E+01
0. 237500E+02	0. 777203E+00	0. 414547E+01	0. 100000E+01	0. 611760E+01
0. 240000E+02	0. 777229E+00	0. 414549E+01	0. 100000E+01	0. 611763E+01
0. 242500E+02	0. 777254E+00	0. 414550E+01	0. 100000E+01	0. 611765E+01
0. 245000E+02	0. 777278E+00	0. 414551E+01	0. 100000E+01	0. 611767E+01
0. 247500E+02	0. 777301E+00	0. 414552E+01	0. 100000E+01	0. 611769E+01

TABLE 6-3. SPECTRAL RADIUS AND NORMS BY WILSON θ AND NEWMARK β METHODS FOR $\xi = 0$

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0. 000000E+00	0. 100000E+01	0. 176762E+01	0. 100000E+01	0. 166270E+01
0. 250000E+00	0. 890956E+00	0. 196407E+01	0. 926668E+00	0. 223012E+01
0. 500000E+00	0. 731547E+00	0. 283365E+01	0. 867390E+00	0. 381794E+01
0. 750000E+00	0. 649282E+00	0. 318161E+01	0. 843953E+00	0. 446412E+01
0. 100000E+01	0. 607216E+00	0. 333084E+01	0. 833645E+00	0. 474643E+01
0. 125000E+01	0. 583818E+00	0. 340579E+01	0. 828394E+00	0. 488958E+01
0. 150000E+01	0. 569739E+00	0. 344821E+01	0. 825399E+00	0. 497101E+01
0. 175000E+01	0. 560703E+00	0. 347438E+01	0. 823541E+00	0. 502142E+01
0. 200000E+01	0. 554594E+00	0. 349162E+01	0. 822314E+00	0. 505469E+01
0. 225000E+01	0. 550287E+00	0. 350356E+01	0. 821462E+00	0. 507775E+01
0. 250000E+01	0. 547144E+00	0. 351215E+01	0. 820848E+00	0. 509437E+01
0. 275000E+01	0. 544784E+00	0. 351854E+01	0. 820391E+00	0. 510674E+01
0. 300000E+01	0. 542969E+00	0. 352342E+01	0. 820042E+00	0. 511619E+01
0. 325000E+01	0. 541544E+00	0. 352723E+01	0. 819769E+00	0. 512357E+01
0. 350000E+01	0. 540406E+00	0. 353026E+01	0. 819552E+00	0. 512944E+01
0. 375000E+01	0. 539483E+00	0. 353270E+01	0. 819377E+00	0. 513418E+01
0. 400000E+01	0. 538723E+00	0. 353471E+01	0. 819233E+00	0. 513807E+01
0. 425000E+01	0. 538092E+00	0. 353637E+01	0. 819114E+00	0. 514130E+01
0. 450000E+01	0. 537561E+00	0. 353777E+01	0. 819013E+00	0. 514400E+01
0. 475000E+01	0. 537111E+00	0. 353895E+01	0. 818928E+00	0. 514630E+01
0. 500000E+01	0. 536725E+00	0. 353996E+01	0. 818856E+00	0. 514626E+01
0. 525000E+01	0. 536393E+00	0. 354083E+01	0. 818793E+00	0. 514995E+01
0. 550000E+01	0. 536104E+00	0. 354159E+01	0. 818739E+00	0. 515141E+01
0. 575000E+01	0. 535852E+00	0. 354225E+01	0. 818692E+00	0. 515269E+01
0 600000E+01	0. 535631E+00	0. 354282E+01	0. 818650E+00	0. 515381E+01
0. 625000E+01	0. 535435E+00	0. 354333E+01	0. 818614E+00	0. 515480E+01
0. 650000E+01	0. 535261E+00	0. 354379E+01	0. 818581E+00	0. 515568E+01
0. 675000E+01	0. 535106E+00	0. 354419E+01	0. 818552E+00	0. 515646E+01
0. 700000E+01	0. 534967E+00	0. 354455E+01	0. 818526E+00	0. 515716E+01
0. 725000E+01	0. 534843E+00	0. 354488E+01	0. 818503E+00	0. 515779E+01
0. 750000E+01	0. 534730E+00	0. 354517E+01	0. 818482E+00	0. 515836E+01
0. 775000E+01	0. 534628E+00	0. 354543E+01	0. 818463E+00	0. 515888E+01
0. 800000E+01	0. 534536E+00	0. 354567E+01	0. 818446E+00	0. 515934E+01
0. 825000E+01	0. 534452E+00	0. 354589E+01	0. 818430E+00	0. 515977E+01
0. 850000E+01	0. 534375E+00	0. 354609E+01	0. 818416E+00	0. 516016E+01
0. 875000E+01	0. 534304E+00	0. 354628E+01	0. 818402E+00	0. 516051E+01
0. 900000E+01	0. 534239E+00	0. 354644E+01	0. 818390E+00	0. 516084E+01
0. 925000E+01	0. 534180E+00	0. 354660E+01	0. 818379E+00	0. 516114E+01
0. 950000E+01	0. 534125E+00	0. 354674E+01	0. 818369E+00	0. 516141E+01
0. 975000E+01	0. 534074E+00	0. 354687E+01	0. 818360E+00	0. 516167E+01
0. 100000E+02	0. 534027E+00	0. 354699E+01	0. 818351E+00	0. 516191E+01
0. 102500E+02	0. 533983E+00	0. 354711E+01	0. 818343E+00	0. 516213E+01
0. 105000E+02	0. 533943E+00	0. 354721E+01	0. 818335E+00	0. 516233E+01
0. 107500E+02	0. 533905E+00	0. 354731E+01	0. 818328E+00	0. 516252E+01
0. 110000E+02	0. 533870E+00	0. 354740E+01	0. 818322E+00	0. 516270E+01
0. 112500E+02	0. 533837E+00	0. 354749E+01	0. 818315E+00	0. 516286E+01
0. 115000E+02	0. 533806E+00	0. 354757E+01	0. 818310E+00	0. 516302E+01
0. 117500E+02	0. 533777E+00	0. 354764E+01	0. 818304E+00	0. 516316E+01
0. 120000E+02	0. 533750E+00	0. 354771E+01	0. 818299E+00	0. 516330E+01
0. 122500E+02	0. 533724E+00	0. 354778E+01	0. 818294E+00	0. 516343E+01
0. 125000E+02	0. 533701E+00	0. 354784E+01	0. 818290E+00	0. 516355E+01

NOTE CONSTANT VALUES ARE:

$\beta = 0.3025$ $\quad \theta = 1.5$

$\gamma = 0.6$ $\quad \xi = 0$

TABLE 6-3. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0.127500E+02	0.533678E+00	0.354790E+01	0.818286E+00	0.516366E+01
0.130000E+02	0.533657E+00	0.354795E+01	0.818282E+00	0.516377E+01
0.132500E+02	0.533637E+00	0.354801E+01	0.818278E+00	0.516387E+01
0.135000E+02	0.533618E+00	0.354806E+01	0.818275E+00	0.516397E+01
0.137500E+02	0.533600E+00	0.354810E+01	0.818271E+00	0.516406E+01
0.140000E+02	0.533583E+00	0.354815E+01	0.818268E+00	0.516414E+01
0.142500E+02	0.533567E+00	0.354819E+01	0.818265E+00	0.516422E+01
0.145000E+02	0.533551E+00	0.354823E+01	0.818262E+00	0.516430E+01
0.147500E+02	0.533537E+00	0.354826E+01	0.818260E+00	0.516437E+01
0.150000E+02	0.533523E+00	0.354830E+01	0.818257E+00	0.516444E+01
0.152500E+02	0.533510E+00	0.354833E+01	0.818255E+00	0.516451E+01
0.155000E+02	0.533497E+00	0.354837E+01	0.818252E+00	0.516457E+01
0.157500E+02	0.533485E+00	0.354840E+01	0.818250E+00	0.516463E+01
0.160000E+02	0.533474E+00	0.354843E+01	0.818248E+00	0.516469E+01
0.162500E+02	0.533463E+00	0.354845E+01	0.818246E+00	0.516474E+01
0.165000E+02	0.533453E+00	0.354848E+01	0.818244E+00	0.516479E+01
0.167500E+02	0.533443E+00	0.354851E+01	0.818242E+00	0.516484E+01
0.170000E+02	0.533434E+00	0.354853E+01	0.818240E+00	0.516489E+01
0.172500E+02	0.533425E+00	0.354856E+01	0.818239E+00	0.516494E+01
0.175000E+02	0.533416E+00	0.354858E+01	0.818237E+00	0.516498E+01
0.177500E+02	0.533408E+00	0.354860E+01	0.818235E+00	0.516502E+01
0.180000E+02	0.533400E+00	0.354862E+01	0.818234E+00	0.516506E+01
0.182500E+02	0.533392E+00	0.354864E+01	0.818233E+00	0.516510E+01
0.185000E+02	0.533385E+00	0.354866E+01	0.818231E+00	0.516514E+01
0.187500E+02	0.533378E+00	0.354868E+01	0.818230E+00	0.516517E+01
0.190000E+02	0.533371E+00	0.354869E+01	0.818229E+00	0.516521E+01
0.192500E+02	0.533364E+00	0.354871E+01	0.818227E+00	0.516524E+01
0.195000E+02	0.533358E+00	0.354873E+01	0.818226E+00	0.516527E+01
0.197500E+02	0.533352E+00	0.354874E+01	0.818225E+00	0.516530E+01
0.200000E+02	0.533346E+00	0.354876E+01	0.818224E+00	0.516533E+01
0.202500E+02	0.533341E+00	0.354877E+01	0.818223E+00	0.516536E+01
0.205000E+02	0.533335E+00	0.354879E+01	0.818222E+00	0.516539E+01
0.207500E+02	0.533330E+00	0.354880E+01	0.818221E+00	0.516541E+01
0.210000E+02	0.533325E+00	0.354881E+01	0.818220E+00	0.516544E+01
0.212500E+02	0.533320E+00	0.354882E+01	0.818219E+00	0.516546E+01
0.215000E+02	0.533316E+00	0.354884E+01	0.818218E+00	0.516548E+01
0.217500E+02	0.533311E+00	0.354885E+01	0.818218E+00	0.516551E+01
0.220000E+02	0.533307E+00	0.354886E+01	0.818217E+00	0.516553E+01
0.222500E+02	0.533303E+00	0.354887E+01	0.818216E+00	0.516555E+01
0.225000E+02	0.533299E+00	0.354888E+01	0.818215E+00	0.516557E+01
0.227500E+02	0.533295E+00	0.354889E+01	0.818214E+00	0.516559E+01
0.230000E+02	0.533291E+00	0.354890E+01	0.818214E+00	0.516561E+01
0.232500E+02	0.533287E+00	0.354891E+01	0.818213E+00	0.516563E+01
0.235000E+02	0.533284E+00	0.354892E+01	0.818212E+00	0.516564E+01
0.237500E+02	0.533280E+00	0.354893E+01	0.818212E+00	0.516566E+01
0.240000E+02	0.533277E+00	0.354894E+01	0.818211E+00	0.516568E+01
0.242500E+02	0.533274E+00	0.354895E+01	0.818211E+00	0.516570E+01
0.245000E+02	0.533270E+00	0.354895E+01	0.818210E+00	0.516571E+01
0.247500E+02	0.533267E+00	0.354896E+01	0.818209E+00	0.516573E+01

TABLE 6-4. SPECTRAL RADIUS AND NORMS BY WILSON θ AND NEWMARK β METHODS FOR $\xi = 0$

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0.000000E+00	0.100000E+01	0.178535E+01	0.100000E+01	0.162712E+01
0.250000E+00	0.864289E+00	0.181064E+01	0.798490E+00	0.197470E+01
0.500000E+00	0.715152E+00	0.233340E+01	0.653671E+00	0.309811E+01
0.750000E+00	0.651198E+00	0.253185E+01	0.598609E+00	0.349840E+01
0.100000E+01	0.621652E+00	0.261503E+01	0.574546E+00	0.366474E+01
0.125000E+01	0.606167E+00	0.265637E+01	0.562298E+00	0.374722E+01
0.150000E+01	0.597184E+00	0.267963E+01	0.555311E+00	0.379360E+01
0.175000E+01	0.591553E+00	0.269394E+01	0.550976E+00	0.382212E+01
0.200000E+01	0.587808E+00	0.270335E+01	0.548112E+00	0.384087E+01
0.225000E+01	0.585197E+00	0.270985E+01	0.546124E+00	0.385382E+01
0.250000E+01	0.583307E+00	0.271453E+01	0.544690E+00	0.386314E+01
0.275000E+01	0.581896E+00	0.271801E+01	0.543623E+00	0.387006E+01
0.300000E+01	0.580817E+00	0.272066E+01	0.542807E+00	0.387534E+01
0.325000E+01	0.579972E+00	0.272273E+01	0.542170E+00	0.387946E+01
0.350000E+01	0.579300E+00	0.272438E+01	0.541663E+00	0.388274E+01
0.375000E+01	0.578755E+00	0.272570E+01	0.541253E+00	0.388539E+01
0.400000E+01	0.578308E+00	0.272679E+01	0.540917E+00	0.388756E+01
0.425000E+01	0.577937E+00	0.272770E+01	0.540638E+00	0.388936E+01
0.450000E+01	0.577626E+00	0.272845E+01	0.540404E+00	0.389087E+01
0.475000E+01	0.577362E+00	0.272910E+01	0.540206E+00	0.389214E+01
0.500000E+01	0.577136E+00	0.272965E+01	0.540037E+00	0.389324E+01
0.525000E+01	0.576942E+00	0.273012E+01	0.539891E+00	0.389418E+01
0.550000E+01	0.576773E+00	0.273053E+01	0.539764E+00	0.389499E+01
0.575000E+01	0.576626E+00	0.273088E+01	0.539654E+00	0.389570E+01
0.600000E+01	0.576496E+00	0.273120E+01	0.539557E+00	0.389633E+01
0.625000E+01	0.576382E+00	0.273147E+01	0.539471E+00	0.389688E+01
0.650000E+01	0.576281E+00	0.273172E+01	0.539395E+00	0.389737E+01
0.675000E+01	0.576191E+00	0.273194E+01	0.539327E+00	0.389780E+01
0.700000E+01	0.576110E+00	0.273213E+01	0.539267E+00	0.389819E+01
0.725000E+01	0.576037E+00	0.273231E+01	0.539212E+00	0.389854E+01
0.750000E+01	0.575971E+00	0.273247E+01	0.539163E+00	0.389886E+01
0.775000E+01	0.575912E+00	0.273261E+01	0.539119E+00	0.389915E+01
0.800000E+01	0.575858E+00	0.273274E+01	0.539078E+00	0.389941E+01
0.825000E+01	0.575809E+00	0.273286E+01	0.539042E+00	0.389964E+01
0.850000E+01	0.575764E+00	0.273297E+01	0.539008E+00	0.389986E+01
0.875000E+01	0.575723E+00	0.273307E+01	0.538977E+00	0.390006E+01
0.900000E+01	0.575686E+00	0.273316E+01	0.538949E+00	0.390024E+01
0.925000E+01	0.575651E+00	0.273325E+01	0.538923E+00	0.390040E+01
0.950000E+01	0.575619E+00	0.273332E+01	0.538899E+00	0.390056E+01
0.975000E+01	0.575589E+00	0.273339E+01	0.538877E+00	0.390070E+01
0.100000E+02	0.575562E+00	0.273346E+01	0.538856E+00	0.390083E+01
0.102500E+02	0.575537E+00	0.273352E+01	0.538837E+00	0.390095E+01
0.105000E+02	0.575513E+00	0.273358E+01	0.538820E+00	0.390107E+01
0.107500E+02	0.575491E+00	0.273363E+01	0.538803E+00	0.390117E+01
0.110000E+02	0.575471E+00	0.273368E+01	0.538788E+00	0.390127E+01
0.112500E+02	0.575452E+00	0.273373E+01	0.538774E+00	0.390136E+01
0.115000E+02	0.575434E+00	0.273377E+01	0.538760E+00	0.390145E+01
0.117500E+02	0.575417E+00	0.273381E+01	0.538748E+00	0.390153E+01
0.120000E+02	0.575401E+00	0.273385E+01	0.538736E+00	0.390161E+01
0.122500E+02	0.575386E+00	0.273389E+01	0.538725E+00	0.390168E+01
0.125000E+02	0.575372E+00	0.273392E+01	0.538714E+00	0.390175E+01

NOTE CONSTANT VALUES ARE:

$\beta = 0.4225$ $\theta = 1.7$

$\gamma = 0.8$ $\xi = 0$

TABLE 6-4. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0.127500E+02	0.575359E+00	0.273395E+01	0.538705E+00	0.390181E+01
0.130000E+02	0.575347E+00	0.273398E+01	0.538695E+00	0.390187E+01
0.132500E+02	0.575335E+00	0.273401E+01	0.538687E+00	0.390192E+01
0.135000E+02	0.575324E+00	0.273404E+01	0.538678E+00	0.390198E+01
0.137500E+02	0.575314E+00	0.273406E+01	0.538671E+00	0.390203E+01
0.140000E+02	0.575304E+00	0.273408E+01	0.538663E+00	0.390208E+01
0.142500E+02	0.575295E+00	0.273411E+01	0.538656E+00	0.390212E+01
0.145000E+02	0.575286E+00	0.273413E+01	0.538649E+00	0.390216E+01
0.147500E+02	0.575277E+00	0.273415E+01	0.538643E+00	0.390220E+01
0.150000E+02	0.575269E+00	0.273417E+01	0.538637E+00	0.390224E+01
0.152500E+02	0.575262E+00	0.273419E+01	0.538631E+00	0.390228E+01
0.155000E+02	0.575254E+00	0.273420E+01	0.538626E+00	0.390231E+01
0.157500E+02	0.575247E+00	0.273422E+01	0.538621E+00	0.390235E+01
0.160000E+02	0.575241E+00	0.273424E+01	0.538616E+00	0.390238E+01
0.162500E+02	0.575235E+00	0.273425E+01	0.538611E+00	0.390241E+01
0.165000E+02	0.575229E+00	0.273427E+01	0.538607E+00	0.390244E+01
0.167500E+02	0.575223E+00	0.273428E+01	0.538602E+00	0.390247E+01
0.170000E+02	0.575217E+00	0.273429E+01	0.538598E+00	0.390249E+01
0.172500E+02	0.575212E+00	0.273431E+01	0.538594E+00	0.390252E+01
0.175000E+02	0.575207E+00	0.273432E+01	0.538591E+00	0.390254E+01
0.177500E+02	0.575202E+00	0.273433E+01	0.538587E+00	0.390256E+01
0.180000E+02	0.575198E+00	0.273434E+01	0.538584E+00	0.390259E+01
0.182500E+02	0.575193E+00	0.273435E+01	0.538580E+00	0.390261E+01
0.185000E+02	0.575189E+00	0.273436E+01	0.538577E+00	0.390263E+01
0.187500E+02	0.575185E+00	0.273437E+01	0.538574E+00	0.390265E+01
0.190000E+02	0.575181E+00	0.273438E+01	0.538571E+00	0.390267E+01
0.192500E+02	0.575177E+00	0.273439E+01	0.538568E+00	0.390269E+01
0.195000E+02	0.575173E+00	0.273440E+01	0.538565E+00	0.390270E+01
0.197500E+02	0.575170E+00	0.273441E+01	0.538563E+00	0.390272E+01
0.200000E+02	0.575167E+00	0.273442E+01	0.538560E+00	0.390274E+01
0.202500E+02	0.575163E+00	0.273442E+01	0.538558E+00	0.390275E+01
0.205000E+02	0.575160E+00	0.273443E+01	0.538556E+00	0.390277E+01
0.207500E+02	0.575157E+00	0.273444E+01	0.538553E+00	0.390278E+01
0.210000E+02	0.575154E+00	0.273445E+01	0.538551E+00	0.390280E+01
0.212500E+02	0.575152E+00	0.273445E+01	0.538549E+00	0.390281E+01
0.215000E+02	0.575149E+00	0.273446E+01	0.538547E+00	0.390282E+01
0.217500E+02	0.575146E+00	0.273447E+01	0.538545E+00	0.390283E+01
0.220000E+02	0.575144E+00	0.273447E+01	0.538543E+00	0.390285E+01
0.222500E+02	0.575141E+00	0.273448E+01	0.538541E+00	0.390286E+01
0.225000E+02	0.575139E+00	0.273448E+01	0.538540E+00	0.390287E+01
0.227500E+02	0.575137E+00	0.273449E+01	0.538538E+00	0.390288E+01
0.230000E+02	0.575134E+00	0.273449E+01	0.538536E+00	0.390289E+01
0.232500E+02	0.575132E+00	0.273450E+01	0.538535E+00	0.390290E+01
0.235000E+02	0.575130E+00	0.273450E+01	0.538533E+00	0.390291E+01
0.237500E+02	0.575128E+00	0.273451E+01	0.538532E+00	0.390292E+01
0.240000E+02	0.575126E+00	0.273451E+01	0.538530E+00	0.390293E+01
0.242500E+02	0.575124E+00	0.273452E+01	0.538529E+00	0.390294E+01
0.245000E+02	0.575123E+00	0.273452E+01	0.538527E+00	0.390295E+01
0.247500E+02	0.575121E+00	0.273453E+01	0.538526E+00	0.390296E+01

TABLE 6-5. SPECTRAL RADIUS AND NORMS BY WILSON θ AND NEWMARK β METHODS FOR $\xi = 0$

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0. 000000E+00	0. 100000E+01	0. 180773E+01	0. 100000E+01	0. 162712E+01
0. 250000E+00	0. 840120E+00	0. 167656E+01	0. 652682E+00	0. 170049E+01
0. 500000E+00	0. 719332E+00	0. 190876E+01	0. 436617E+00	0. 241699E+01
0. 750000E+00	0. 677452E+00	0. 199299E+01	0. 352318E+00	0. 263880E+01
0. 100000E+01	0. 659864E+00	0. 202761E+01	0. 313356E+00	0. 272677E+01
0. 125000E+01	0. 651079E+00	0. 204468E+01	0. 292678E+00	0. 276955E+01
0. 150000E+01	0. 646119E+00	0. 205424E+01	0. 280546E+00	0. 279335E+01
0. 175000E+01	0. 643060E+00	0. 206010E+01	0. 272876E+00	0. 280791E+01
0. 200000E+01	0. 641046E+00	0. 206395E+01	0. 267741E+00	0. 281743E+01
0. 225000E+01	0. 639652E+00	0. 206661E+01	0. 264145E+00	0. 282400E+01
0. 250000E+01	0. 638649E+00	0. 206852E+01	0. 261533E+00	0. 282872E+01
0. 275000E+01	0. 637903E+00	0. 206994E+01	0. 259578E+00	0. 283222E+01
0. 300000E+01	0. 637334E+00	0. 207102E+01	0. 258079E+00	0. 283489E+01
0. 325000E+01	0. 636889E+00	0. 207187E+01	0. 256904E+00	0. 283697E+01
0. 350000E+01	0. 636536E+00	0. 207254E+01	0. 255967E+00	0. 283862E+01
0. 375000E+01	0. 636250E+00	0. 207308E+01	0. 255208E+00	0. 283996E+01
0. 400000E+01	0. 636016E+00	0. 207352E+01	0. 254585E+00	0. 284105E+01
0. 425000E+01	0. 635822E+00	0. 207389E+01	0. 254066E+00	0. 284196E+01
0. 450000E+01	0. 635659E+00	0. 207420E+01	0. 253631E+00	0. 284272E+01
0. 475000E+01	0. 635521E+00	0. 207446E+01	0. 253262E+00	0. 284336E+01
0. 500000E+01	0. 635403E+00	0. 207469E+01	0. 252946E+00	0. 284391E+01
0. 525000E+01	0. 635302E+00	0. 207488E+01	0. 252674E+00	0. 284439E+01
0. 550000E+01	0. 635214E+00	0. 207504E+01	0. 252438E+00	0. 284480E+01
0. 575000E+01	0. 635137E+00	0. 207519E+01	0. 252232E+00	0. 284515E+01
0. 600000E+01	0. 635069E+00	0. 207532E+01	0. 252051E+00	0. 284547E+01
0. 625000E+01	0. 635010E+00	0. 207543E+01	0. 251891E+00	0. 284575E+01
0. 650000E+01	0. 634957E+00	0. 207553E+01	0. 251749E+00	0. 284599E+01
0. 675000E+01	0. 634910E+00	0. 207562E+01	0. 251622E+00	0. 284621E+01
0. 700000E+01	0. 634868E+00	0. 207570E+01	0. 251509E+00	0. 284641E+01
0. 725000E+01	0. 634830E+00	0. 207577E+01	0. 251407E+00	0. 284658E+01
0. 750000E+01	0. 634796E+00	0. 207584E+01	0. 251315E+00	0. 284674E+01
0. 775000E+01	0. 634765E+00	0. 207589E+01	0. 251232E+00	0. 284689E+01
0. 800000E+01	0. 634737E+00	0. 207595E+01	0. 251156E+00	0. 284702E+01
0. 825000E+01	0. 634711E+00	0. 207600E+01	0. 251087E+00	0. 284714E+01
0. 850000E+01	0. 634688E+00	0. 207604E+01	0. 251024E+00	0. 284725E+01
0. 875000E+01	0. 634667E+00	0. 207608E+01	0. 250967E+00	0. 284735E+01
0. 900000E+01	0. 634647E+00	0. 207612E+01	0. 250914E+00	0. 284744E+01
0. 925000E+01	0. 634629E+00	0. 207615E+01	0. 250865E+00	0. 284752E+01
0. 950000E+01	0. 634612E+00	0. 207618E+01	0. 250821E+00	0. 284760E+01
0. 975000E+01	0. 634597E+00	0. 207621E+01	0. 250779E+00	0. 284767E+01
0. 100000E+02	0. 634583E+00	0. 207624E+01	0. 250741E+00	0. 284774E+01
0. 102500E+02	0. 634569E+00	0. 207626E+01	0. 250705E+00	0. 284780E+01
0. 105000E+02	0. 634557E+00	0. 207629E+01	0. 250672E+00	0. 284785E+01
0. 107500E+02	0. 634546E+00	0. 207631E+01	0. 250641E+00	0. 284791E+01
0. 110000E+02	0. 634535E+00	0. 207633E+01	0. 250612E+00	0. 284796E+01
0. 112500E+02	0. 634525E+00	0. 207635E+01	0. 250585E+00	0. 284800E+01
0. 115000E+02	0. 634516E+00	0. 207637E+01	0 250560E+00	0. 284805E+01
0. 117500E+02	0. 634507E+00	0. 207638E+01	0. 250537E+00	0. 284809E+01
0. 120000E+02	0. 634499E+00	0. 207640E+01	0. 250515E+00	0. 284813E+01
0. 122500E+02	0 634491E+00	0. 207641E+01	0 250494E+00	0. 284816E+01
0. 125000E+02	0. 634484E+00	0. 207643E+01	0 250474E+00	0. 284820E+01

NOTE CONSTANT VALUES ARE:

$\beta = 0.64$ $\theta = 2.0$

$\gamma = 1.1$ $\xi = 0$

TABLE 6-5. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0. 127500E+02	0. 634477E+00	0. 207644E+01	0. 250456E+00	0. 284823E+01
0. 130000E+02	0. 634471E+00	0. 207645E+01	0. 250439E+00	0. 284826E+01
0. 132500E+02	0. 634465E+00	0. 207646E+01	0. 250422E+00	0. 284829E+01
0. 135000E+02	0. 634459E+00	0. 207647E+01	0. 250407E+00	0. 284831E+01
0. 137500E+02	0. 634453E+00	0. 207648E+01	0. 250392E+00	0. 284834E+01
0. 140000E+02	0. 634448E+00	0. 207649E+01	0. 250378E+00	0. 284836E+01
0. 142500E+02	0. 634443E+00	0. 207650E+01	0. 250365E+00	0. 284838E+01
0. 145000E+02	0. 634439E+00	0. 207651E+01	0. 250353E+00	0. 284841E+01
0. 147500E+02	0. 634434E+00	0. 207652E+01	0. 250341E+00	0. 284843E+01
0. 150000E+02	0. 634430E+00	0. 207653E+01	0. 250330E+00	0. 284844E+01
0. 152500E+02	0. 634426E+00	0. 207654E+01	0. 250319E+00	0. 284846E+01
0. 155000E+02	0. 634423E+00	0. 207654E+01	0. 250309E+00	0. 284848E+01
0. 157500E+02	0. 634419E+00	0. 207655E+01	0. 250299E+00	0. 284850E+01
0. 160000E+02	0. 634416E+00	0. 207656E+01	0. 250290E+00	0. 284851E+01
0. 162500E+02	0. 634412E+00	0. 207656E+01	0. 250281E+00	0. 284853E+01
0. 165000E+02	0. 634409E+00	0. 207657E+01	0. 250272E+00	0. 284854E+01
0. 167500E+02	0. 634406E+00	0. 207657E+01	0. 250264E+00	0. 284856E+01
0. 170000E+02	0. 634403E+00	0. 207658E+01	0. 250257E+00	0. 284857E+01
0. 172500E+02	0. 634401E+00	0. 207658E+01	0. 250249E+00	0. 284858E+01
0. 175000E+02	0. 634398E+00	0. 207659E+01	0. 250242E+00	0. 284860E+01
0. 177500E+02	0. 634395E+00	0. 207659E+01	0. 250235E+00	0. 284861E+01
0. 180000E+02	0. 634393E+00	0. 207660E+01	0. 250229E+00	0. 284862E+01
0. 182500E+02	0. 634391E+00	0. 207660E+01	0. 250223E+00	0. 284863E+01
0. 185000E+02	0. 634389E+00	0. 207661E+01	0. 250217E+00	0. 284864E+01
0. 187500E+02	0. 634386E+00	0. 207661E+01	0. 250211E+00	0. 284865E+01
0. 190000E+02	0. 634384E+00	0. 207661E+01	0. 250205E+00	0. 284866E+01
0. 192500E+02	0. 634382E+00	0. 207662E+01	0. 250200E+00	0. 284867E+01
0. 195000E+02	0. 634381E+00	0. 207662E+01	0. 250195E+00	0. 284868E+01
0. 197500E+02	0. 634379E+00	0. 207663E+01	0. 250190E+00	0. 284869E+01
0. 200000E+02	0. 634377E+00	0. 207663E+01	0. 250185E+00	0. 284869E+01
0. 202500E+02	0. 634375E+00	0. 207663E+01	0. 250181E+00	0. 284870E+01
0. 205000E+02	0. 634374E+00	0. 207664E+01	0. 250177E+00	0. 284871E+01
0. 207500E+02	0. 634372E+00	0. 207664E+01	0. 250172E+00	0. 284872E+01
0. 210000E+02	0. 634371E+00	0. 207664E+01	0. 250168E+00	0. 284872E+01
0. 212500E+02	0. 634369E+00	0. 207664E+01	0. 250164E+00	0. 284873E+01
0. 215000E+02	0. 634368E+00	0. 207665E+01	0. 250160E+00	0. 284874E+01
0. 217500E+02	0. 634366E+00	0. 207665E+01	0. 250157E+00	0. 284874E+01
0. 220000E+02	0. 634365E+00	0. 207665E+01	0. 250153E+00	0. 284875E+01
0. 222500E+02	0. 634364E+00	0. 207665E+01	0. 250150E+00	0. 284875E+01
0. 225000E+02	0. 634363E+00	0. 207666E+01	0. 250147E+00	0. 284876E+01
0. 227500E+02	0. 634361E+00	0. 207666E+01	0. 250143E+00	0. 284877E+01
0. 230000E+02	0. 634360E+00	0. 207666E+01	0. 250140E+00	0. 284877E+01
0. 232500E+02	0. 634359E+00	0. 207666E+01	0. 250137E+00	0. 284878E+01
0. 235000E+02	0. 634358E+00	0. 207666E+01	0. 250134E+00	0. 284878E+01
0. 237500E+02	0. 634357E+00	0. 207667E+01	0. 250132E+00	0. 284879E+01
0. 240000E+02	0. 634356E+00	0. 207667E+01	0. 250129E+00	0. 284879E+01
0. 242500E+02	0. 634355E+00	0. 207667E+01	0. 250126E+00	0. 284880E+01
0. 245000E+02	0. 634354E+00	0. 207667E+01	0. 250124E+00	0. 284880E+01
0. 247500E+02	0. 634353E+00	0. 207667E+01	0. 250121E+00	0. 284880E+01

TABLE 6-6. SPECTRAL RADIUS AND NORMS BY WILSON θ AND NEWMARK β METHODS FOR $\xi = 0$

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0. 000000E+00	0. 100000E+01	0. 183153E+01	0. 100000E+01	0. 175963E+01
0. 250000E+00	0. 827830E+00	0. 160266E+01	0. 537029E+00	0. 145762E+01
0. 500000E+00	0. 742382E+00	0. 166932E+01	0. 303314E+00	0. 189843E+01
0. 750000E+00	0. 717805E+00	0. 169374E+01	0. 207584E+00	0. 202004E+01
0. 100000E+01	0. 708155E+00	0. 170370E+01	0. 157177E+00	0. 206667E+01
0. 125000E+01	0. 703479E+00	0. 170858E+01	0. 126304E+00	0. 208902E+01
0. 150000E+01	0. 700879E+00	0. 171131E+01	0. 105511E+00	0. 210137E+01
0. 175000E+01	0. 699291E+00	0. 171298E+01	0. 905719E-01	0. 210890E+01
0. 200000E+01	0. 698252E+00	0. 171408E+01	0. 793267E-01	0. 211381E+01
0. 225000E+01	0. 697535E+00	0. 171483E+01	0. 705592E-01	0. 211719E+01
0. 250000E+01	0. 697021E+00	0. 171538E+01	0. 635334E-01	0. 211962E+01
0. 275000E+01	0. 696639E+00	0. 171578E+01	0. 577778E-01	0. 212141E+01
0. 300000E+01	0. 696348E+00	0. 171609E+01	0. 529771E-01	0. 212278E+01
0. 325000E+01	0. 696122E+00	0. 171633E+01	0. 489121E-01	0. 212385E+01
0. 350000E+01	0. 695942E+00	0. 171652E+01	0. 454259E-01	0. 212470E+01
0. 375000E+01	0. 695796E+00	0. 171668E+01	0. 424031E-01	0. 212538E+01
0. 400000E+01	0. 695677E+00	0. 171680E+01	0. 397573E-01	0. 212594E+01
0. 425000E+01	0. 695578E+00	0. 171691E+01	0. 374220E-01	0. 212641E+01
0. 450000E+01	0. 695495E+00	0. 171699E+01	0. 353457E-01	0. 212680E+01
0. 475000E+01	0. 695425E+00	0. 171707E+01	0. 334875E-01	0. 212713E+01
0. 500000E+01	0. 695365E+00	0. 171713E+01	0. 318149E-01	0. 212741E+01
0. 525000E+01	0. 695314E+00	0. 171719E+01	0. 303013E-01	0. 212765E+01
0. 550000E+01	0. 695269E+00	0. 171723E+01	0. 289252E-01	0. 212786E+01
0. 575000E+01	0. 695230E+00	0. 171728E+01	0. 276685E-01	0. 212805E+01
0. 600000E+01	0. 695196E+00	0. 171731E+01	0. 265165E-01	0. 212821E+01
0. 625000E+01	0. 695165E+00	0. 171734E+01	0. 254565E-01	0. 212835E+01
0. 650000E+01	0. 695138E+00	0. 171737E+01	0. 244780E-01	0. 212847E+01
0. 675000E+01	0. 695115E+00	0. 171740E+01	0. 235720E-01	0. 212859E+01
0. 700000E+01	0. 695093E+00	0. 171742E+01	0. 227305E-01	0. 212869E+01
0. 725000E+01	0. 695074E+00	0. 171744E+01	0. 219471E-01	0. 212878E+01
0. 750000E+01	0. 695057E+00	0. 171746E+01	0. 212159E-01	0. 212886E+01
0. 775000E+01	0. 695041E+00	0. 171748E+01	0. 205318E-01	0. 212893E+01
0. 800000E+01	0. 695027E+00	0. 171749E+01	0. 198904E-01	0. 212900E+01
0. 825000E+01	0. 695014E+00	0. 171750E+01	0. 192879E-01	0. 212906E+01
0. 850000E+01	0. 695002E+00	0. 171752E+01	0. 187208E-01	0. 212912E+01
0. 875000E+01	0. 694991E+00	0. 171753E+01	0. 181861E-01	0. 212917E+01
0. 900000E+01	0. 694981E+00	0. 171754E+01	0. 176811E-01	0. 212921E+01
0. 925000E+01	0. 694972E+00	0. 171755E+01	0. 172034E-01	0. 212926E+01
0. 950000E+01	0. 694964E+00	0. 171756E+01	0. 167508E-01	0. 212930E+01
0. 975000E+01	0. 694956E+00	0. 171757E+01	0. 163214E-01	0. 212933E+01
0. 100000E+02	0. 694949E+00	0. 171757E+01	0. 159135E-01	0. 212937E+01
0. 102500E+02	0. 694942E+00	0. 171758E+01	0. 155254E-01	0. 212940E+01
0. 105000E+02	0. 694936E+00	0. 171759E+01	0. 151559E-01	0. 212943E+01
0. 107500E+02	0. 694930E+00	0. 171759E+01	0. 148035E-01	0. 212945E+01
0. 110000E+02	0. 694924E+00	0. 171760E+01	0. 144671E-01	0. 212948E+01
0. 112500E+02	0. 694919E+00	0. 171760E+01	0. 141457E-01	0. 212950E+01
0. 115000E+02	0. 694915E+00	0. 171761E+01	0. 138382E-01	0. 212953E+01
0. 117500E+02	0. 694910E+00	0. 171761E+01	0. 135439E-01	0. 212955E+01
0. 120000E+02	0. 694906E+00	0. 171762E+01	0. 132617E-01	0. 212957E+01
0. 122500E+02	0. 694902E+00	0. 171762E+01	0. 129911E-01	0. 212958E+01
0. 125000E+02	0. 694898E+00	0. 171763E+01	0. 127314E-01	0. 212960E+01

NOTE CONSTANT VALUES ARE:

$\beta = 1.0 \quad \theta = 2.4$

$\gamma = 1.5 \quad \xi = 0$

TABLE 6-6. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY WILSON	SPECTRAL NORM BY WILSON	SPECTRAL RADIUS BY NEWMARK	SPECTRAL NORM BY NEWMARK
0.127500E+02	0.694895E+00	0.171763E+01	0.124818E-01	0.212962E+01
0.130000E+02	0.694892E+00	0.171763E+01	0.122418E-01	0.212963E+01
0.132500E+02	0.694889E+00	0.171764E+01	0.120108E-01	0.212965E+01
0.135000E+02	0.694886E+00	0.171764E+01	0.117884E-01	0.212966E+01
0.137500E+02	0.694883E+00	0.171764E+01	0.115741E-01	0.212967E+01
0.140000E+02	0.694880E+00	0.171765E+01	0.113675E-01	0.212969E+01
0.142500E+02	0.694878E+00	0.171765E+01	0.111681E-01	0.212970E+01
0.145000E+02	0.694876E+00	0.171765E+01	0.109755E-01	0.212971E+01
0.147500E+02	0.694873E+00	0.171765E+01	0.107895E-01	0.212972E+01
0.150000E+02	0.694871E+00	0.171766E+01	0.106097E-01	0.212973E+01
0.152500E+02	0.694869E+00	0.171766E+01	0.104358E-01	0.212974E+01
0.155000E+02	0.694867E+00	0.171766E+01	0.102675E-01	0.212975E+01
0.157500E+02	0.694866E+00	0.171766E+01	0.101046E-01	0.212976E+01
0.160000E+02	0.694864E+00	0.171766E+01	0.994669E-02	0.212977E+01
0.162500E+02	0.694862E+00	0.171767E+01	0.979368E-02	0.212977E+01
0.165000E+02	0.694861E+00	0.171767E+01	0.964531E-02	0.212978E+01
0.167500E+02	0.694859E+00	0.171767E+01	0.950136E-02	0.212979E+01
0.170000E+02	0.694858E+00	0.171767E+01	0.936165E-02	0.212979E+01
0.172500E+02	0.694856E+00	0.171767E+01	0.922598E-02	0.212980E+01
0.175000E+02	0.694855E+00	0.171767E+01	0.909419E-02	0.212981E+01
0.177500E+02	0.694854E+00	0.171767E+01	0.896612E-02	0.212981E+01
0.180000E+02	0.694852E+00	0.171768E+01	0.884160E-02	0.212982E+01
0.182500E+02	0.694851E+00	0.171768E+01	0.872049E-02	0.212982E+01
0.185000E+02	0.694850E+00	0.171768E+01	0.860265E-02	0.212983E+01
0.187500E+02	0.694849E+00	0.171768E+01	0.848796E-02	0.212983E+01
0.190000E+02	0.694848E+00	0.171768E+01	0.837628E-02	0.212984E+01
0.192500E+02	0.694847E+00	0.171768E+01	0.826751E-02	0.212984E+01
0.195000E+02	0.694846E+00	0.171768E+01	0.816152E-02	0.212985E+01
0.197500E+02	0.694845E+00	0.171768E+01	0.805822E-02	0.212985E+01
0.200000E+02	0.694844E+00	0.171768E+01	0.795750E-02	0.212986E+01
0.202500E+02	0.694843E+00	0.171769E+01	0.785926E-02	0.212986E+01
0.205000E+02	0.694843E+00	0.171769E+01	0.776342E-02	0.212986E+01
0.207500E+02	0.694842E+00	0.171769E+01	0.766989E-02	0.212987E+01
0.210000E+02	0.694841E+00	0.171769E+01	0.757859E-02	0.212987E+01
0.212500E+02	0.694840E+00	0.171769E+01	0.748943E-02	0.212988E+01
0.215000E+02	0.694840E+00	0.171769E+01	0.740235E-02	0.212988E+01
0.217500E+02	0.694839E+00	0.171769E+01	0.731727E-02	0.212988E+01
0.220000E+02	0.694838E+00	0.171769E+01	0.723413E-02	0.212989E+01
0.222500E+02	0.694838E+00	0.171769E+01	0.715285E-02	0.212989E+01
0.225000E+02	0.694837E+00	0.171769E+01	0.707338E-02	0.212989E+01
0.227500E+02	0.694836E+00	0.171769E+01	0.699565E-02	0.212989E+01
0.230000E+02	0.694836E+00	0.171769E+01	0.691961E-02	0.212990E+01
0.232500E+02	0.694835E+00	0.171769E+01	0.684521E-02	0.212990E+01
0.235000E+02	0.694835E+00	0.171769E+01	0.677240E-02	0.212990E+01
0.237500E+02	0.694834E+00	0.171769E+01	0.670111E-02	0.212990E+01
0.240000E+02	0.694834E+00	0.171770E+01	0.663131E-02	0.212991E+01
0.242500E+02	0.694833E+00	0.171770E+01	0.656295E-02	0.212991E+01
0.245000E+02	0.694833E+00	0.171770E+01	0.649598E-02	0.212991E+01
0.247500E+02	0.694832E+00	0.171770E+01	0.643037E-02	0.212991E+01

TABLE 6-7. SPECTRAL NORM AND RADIUS BY HUGHES METHOD

	β = 0.470074		β = 0.5	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162439E+01	0.100000E+01	0.162286E+01
0.500000E-01	0.941390E+00	0.149436E+01	0.941664E+00	0.149009E+01
0.100000E+00	0.863680E+00	0.133452E+01	0.866210E+00	0.132659E+01
0.200000E+00	0.765916E+00	0.131845E+01	0.777443E+00	0.129433E+01
0.300000E+00	0.723414E+00	0.146359E+01	0.742052E+00	0.141921E+01
0.500000E+00	0.687153E+00	0.167132E+01	0.713663E+00	0.159933E+01
0.750000E+00	0.669184E+00	0.179286E+01	0.700231E+00	0.170391E+01
0.100000E+01	0.660617E+00	0.185282E+01	0.693985E+00	0.175519E+01
0.150000E+01	0.652563E+00	0.190945E+01	0.688218E+00	0.180339E+01
0.200000E+01	0.648795E+00	0.193579E+01	0.685560E+00	0.182572E+01
0.300000E+01	0.645240E+00	0.196043E+01	0.683079E+00	0.184656E+01
0.400000E+01	0.643552E+00	0.197202E+01	0.681911E+00	0.185633E+01
0.500000E+01	0.642570E+00	0.197872E+01	0.681236E+00	0.186198E+01
0.100000E+02	0.640680E+00	0.199151E+01	0.679944E+00	0.187274E+01
0.200000E+02	0.639773E+00	0.199759E+01	0.679328E+00	0.187784E+01
0.300000E+02	0.639477E+00	0.199956E+01	0.679127E+00	0.187950E+01
0.400000E+02	0.639330E+00	0.200054E+01	0.679028E+00	0.188032E+01
0.500000E+02	0.639243E+00	0.200113E+01	0.678969E+00	0.188081E+01
0.750000E+02	0.639126E+00	0.200190E+01	0.678890E+00	0.188146E+01
0.100000E+03	0.639068E+00	0.200229E+01	0.678851E+00	0.188179E+01
0.200000E+03	0.638981E+00	0.200287E+01	0.678792E+00	0.188227E+01
0.300000E+03	0.638952E+00	0.200306E+01	0.678773E+00	0.188243E+01
0.400000E+03	0.638937E+00	0.200316E+01	0.678763E+00	0.188251E+01
0.500000E+03	0.638929E+00	0.200321E+01	0.678757E+00	0.188256E+01
0.750000E+03	0.638917E+00	0.200329E+01	0.678749E+00	0.188263E+01
0.100000E+04	0.638911E+00	0.200333E+01	0.678745E+00	0.188266E+01
0.200000E+04	0.638903E+00	0.200339E+01	0.678739E+00	0.188271E+01
0.300000E+04	0.638900E+00	0.200341E+01	0.678737E+00	0.188272E+01
0.400000E+04	0.638898E+00	0.200342E+01	0.678736E+00	0.188273E+01
0.500000E+04	0.638898E+00	0.200342E+01	0.678736E+00	0.188274E+01
0.750000E+04	0.638896E+00	0.200343E+01	0.678735E+00	0.188274E+01
0.100000E+05	0.638896E+00	0.200343E+01	0.678735E+00	0.188275E+01

NOTE CONSTANT VALUES ARE:

$\alpha = -0.1$ $\theta = 1.4$

$\gamma = 1.39296$ $\xi = 0.1$

TABLE 6-7. (CONT.)

DELTAT/PERIOD	$\beta = 0.6$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.6$ SPECTRAL NORM BY HUGHES	$\beta = 0.7$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.7$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161927E+01	0.100000E+01	0.161803E+01
0.500000E-01	0.942565E+00	0.147730E+01	0.943444E+00	0.146667E+01
0.100000E+00	0.874146E+00	0.130292E+01	0.881358E+00	0.128293E+01
0.200000E+00	0.809720E+00	0.122813E+01	0.834832E+00	0.117848E+01
0.300000E+00	0.791052E+00	0.130303E+01	0.826267E+00	0.122167E+01
0.500000E+00	0.779404E+00	0.141829E+01	0.823461E+00	0.129882E+01
0.750000E+00	0.774922E+00	0.148426E+01	0.823269E+00	0.134314E+01
0.100000E+01	0.773081E+00	0.151610E+01	0.823441E+00	0.136433E+01
0.150000E+01	0.771545E+00	0.154564E+01	0.823778E+00	0.138381E+01
0.200000E+01	0.770902E+00	0.155918E+01	0.824008E+00	0.139265E+01
0.300000E+01	0.770349E+00	0.157170E+01	0.824277E+00	0.140077E+01
0.400000E+01	0.770107E+00	0.157753E+01	0.824427E+00	0.140453E+01
0.500000E+01	0.769974E+00	0.158088E+01	0.824521E+00	0.140669E+01
0.100000E+02	0.769735E+00	0.158724E+01	0.824721E+00	0.141076E+01
0.200000E+02	0.769629E+00	0.159025E+01	0.824826E+00	0.141267E+01
0.300000E+02	0.769595E+00	0.159122E+01	0.824862E+00	0.141329E+01
0.400000E+02	0.769579E+00	0.159170E+01	0.824880E+00	0.141359E+01
0.500000E+02	0.769570E+00	0.159199E+01	0.824891E+00	0.141377E+01
0.750000E+02	0.769557E+00	0.159237E+01	0.824906E+00	0.141402E+01
0.100000E+03	0.769551E+00	0.159256E+01	0.824913E+00	0.141414E+01
0.200000E+03	0.769541E+00	0.159284E+01	0.824924E+00	0.141431E+01
0.300000E+03	0.769538E+00	0.159294E+01	0.824928E+00	0.141437E+01
0.400000E+03	0.769537E+00	0.159298E+01	0.824929E+00	0.141440E+01
0.500000E+03	0.769536E+00	0.159301E+01	0.824931E+00	0.141442E+01
0.750000E+03	0.769535E+00	0.159305E+01	0.824932E+00	0.141445E+01
0.100000E+04	0.769534E+00	0.159307E+01	0.824933E+00	0.141446E+01
0.200000E+04	0.769533E+00	0.159310E+01	0.824934E+00	0.141448E+01
0.300000E+04	0.769533E+00	0.159311E+01	0.824934E+00	0.141448E+01
0.400000E+04	0.769533E+00	0.159311E+01	0.824934E+00	0.141448E+01
0.500000E+04	0.769532E+00	0.159311E+01	0.824935E+00	0.141449E+01
0.750000E+04	0.769532E+00	0.159312E+01	0.824935E+00	0.141449E+01
0.100000E+05	0.769532E+00	0.159312E+01	0.824935E+00	0.141449E+01

TABLE 6-7. (CONT.)

DELTAT/PERIOD	$\beta = 0.8$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.8$ SPECTRAL NORM BY HUGHES	$\beta = 0.916275$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.916275$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161916E+01	0.100000E+01	0.162343E+01
0.500000E-01	0.944302E+00	0.145803E+01	0.945274E+00	0.145030E+01
0.100000E+00	0.887931E+00	0.126593E+01	0.894867E+00	0.124918E+01
0.200000E+00	0.854907E+00	0.114030E+01	0.873683E+00	0.110598E+01
0.300000E+00	0.852765E+00	0.116276E+01	0.876302E+00	0.111287E+01
0.500000E+00	0.854992E+00	0.121684E+01	0.881876E+00	0.115122E+01
0.750000E+00	0.857059E+00	0.124858E+01	0.885331E+00	0.117473E+01
0.100000E+01	0.858250E+00	0.126370E+01	0.887123E+00	0.118593E+01
0.150000E+01	0.859515E+00	0.127751E+01	0.888921E+00	0.119612E+01
0.200000E+01	0.860169E+00	0.128374E+01	0.889814E+00	0.120069E+01
0.300000E+01	0.860833E+00	0.128943E+01	0.890697E+00	0.120484E+01
0.400000E+01	0.861168E+00	0.129205E+01	0.891135E+00	0.120675E+01
0.500000E+01	0.861371E+00	0.129355E+01	0.891396E+00	0.120783E+01
0.100000E+02	0.861777E+00	0.129637E+01	0.891915E+00	0.120987E+01
0.200000E+02	0.861981E+00	0.129768E+01	0.892173E+00	0.121081E+01
0.300000E+02	0.862050E+00	0.129811E+01	0.892258E+00	0.121112E+01
0.400000E+02	0.862084E+00	0.129832E+01	0.892301E+00	0.121127E+01
0.500000E+02	0.862104E+00	0.129844E+01	0.892326E+00	0.121136E+01
0.750000E+02	0.862131E+00	0.129861E+01	0.892361E+00	0.121148E+01
0.100000E+03	0.862145E+00	0.129869E+01	0.892378E+00	0.121154E+01
0.200000E+03	0.862166E+00	0.129881E+01	0.892403E+00	0.121162E+01
0.300000E+03	0.862172E+00	0.129886E+01	0.892412E+00	0.121165E+01
0.400000E+03	0.862176E+00	0.129888E+01	0.892416E+00	0.121167E+01
0.500000E+03	0.862178E+00	0.129889E+01	0.892419E+00	0.121168E+01
0.750000E+03	0.862181E+00	0.129890E+01	0.892422E+00	0.121169E+01
0.100000E+04	0.862182E+00	0.129891E+01	0.892424E+00	0.121169E+01
0.200000E+04	0.862184E+00	0.129892E+01	0.892426E+00	0.121170E+01
0.300000E+04	0.862185E+00	0.129893E+01	0.892427E+00	0.121171E+01
0.400000E+04	0.862185E+00	0.129893E+01	0.892427E+00	0.121171E+01
0.500000E+04	0.862185E+00	0.129893E+01	0.892428E+00	0.121171E+01
0.750000E+04	0.862186E+00	0.129893E+01	0.892428E+00	0.121171E+01
0.100000E+05	0.862186E+00	0.129893E+01	0.892428E+00	0.121171E+01

TABLE 6-8. SPECTRAL NORM AND RADIUS FOR HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 0.5$, $\theta = 1.0$, $\xi = 0.10$

	$\beta = 0.25$		$\beta = 0.50$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.168889E+01	0.100000E+01	0.164104E+01
0.500000E-01	0.969796E+00	0.160876E+01	0.970497E+00	0.152808E+01
0.100000E+00	0.944358E+00	0.152844E+01	0.948833E+00	0.138370E+01
0.200000E+00	0.913620E+00	0.192339E+01	0.932081E+00	0.149110E+01
0.300000E+00	0.904694E+00	0.276424E+01	0.934267E+00	0.186857E+01
0.500000E+00	0.913152E+00	0.407467E+01	0.948395E+00	0.236323E+01
0.750000E+00	0.930483E+00	0.490431E+01	0.961795E+00	0.261973E+01
0.100000E+01	0.943773E+00	0.531061E+01	0.970149E+00	0.273142E+01
0.150000E+01	0.960180E+00	0.567188E+01	0.979458E+00	0.282326E+01
0.200000E+01	0.969423E+00	0.582450E+01	0.984405E+00	0.285974E+01
0.300000E+01	0.979231E+00	0.595246E+01	0.989504E+00	0.288901E+01
0.400000E+01	0.984308E+00	0.600589E+01	0.992099E+00	0.290075E+01
0.500000E+01	0.987398E+00	0.603431E+01	0.993667E+00	0.290684E+01
0.100000E+02	0.993660E+00	0.608253E+01	0.996824E+00	0.291678E+01
0.200000E+02	0.996823E+00	0.610217E+01	0.998410E+00	0.292063E+01
0.300000E+02	0.997880E+00	0.610804E+01	0.998940E+00	0.292174E+01
0.400000E+02	0.998410E+00	0.611085E+01	0.999205E+00	0.292227E+01
0.500000E+02	0.998728E+00	0.611249E+01	0.999364E+00	0.292257E+01
0.750000E+02	0.999152E+00	0.611463E+01	0.999576E+00	0.292296E+01
0.100000E+03	0.999364E+00	0.611568E+01	0.999682E+00	0.292316E+01
0.200000E+03	0.999682E+00	0.611723E+01	0.999841E+00	0.292344E+01
0.300000E+03	0.999788E+00	0.611774E+01	0.999894E+00	0.292353E+01
0.400000E+03	0.999841E+00	0.611799E+01	0.999920E+00	0.292358E+01
0.500000E+03	0.999873E+00	0.611815E+01	0.999936E+00	0.292360E+01
0.750000E+03	0.999915E+00	0.611835E+01	0.999958E+00	0.292364E+01
0.100000E+04	0.999936E+00	0.611845E+01	0.999968E+00	0.292366E+01
0.200000E+04	0.999968E+00	0.611860E+01	0.999984E+00	0.292368E+01
0.300000E+04	0.999979E+00	0.611865E+01	0.999989E+00	0.292369E+01
0.400000E+04	0.999984E+00	0.611867E+01	0.999992E+00	0.292370E+01
0.500000E+04	0.999987E+00	0.611869E+01	0.999994E+00	0.292370E+01
0.750000E+04	0.999992E+00	0.611871E+01	0.999996E+00	0.292370E+01
0.100000E+05	0.999994E+00	0.611872E+01	0.999997E+00	0.292371E+01

TABLE 6-8. (CONT.)

DELTAT/PERIOD	$\beta = 2.00$ SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.206656E+01
0.500000E-01	0.974098E+00	0.173039E+01
0.100000E+00	0.965485E+00	0.130197E+01
0.200000E+00	0.970223E+00	0.117762E+01
0.300000E+00	0.977011E+00	0.119819E+01
0.500000E+00	0.984965E+00	0.123481E+01
0.750000E+00	0.989677E+00	0.125626E+01
0.100000E+01	0.992172E+00	0.126648E+01
0.150000E+01	0.994738E+00	0.127579E+01
0.200000E+01	0.996042E+00	0.127999E+01
0.300000E+01	0.997355E+00	0.128382E+01
0.400000E+01	0.998014E+00	0.128558E+01
0.500000E+01	0.998411E+00	0.128659E+01
0.100000E+02	0.999205E+00	0.128849E+01
0.200000E+02	0.999602E+00	0.128938E+01
0.300000E+02	0.999735E+00	0.128967E+01
0.400000E+02	0.999801E+00	0.128981E+01
0.500000E+02	0.999841E+00	0.128989E+01
0.750000E+02	0.999894E+00	0.129001E+01
0.100000E+03	0.999920E+00	0.129006E+01
0.200000E+03	0.999960E+00	0.129015E+01
0.300000E+03	0.999973E+00	0.129017E+01
0.400000E+03	0.999980E+00	0.129019E+01
0.500000E+03	0.999984E+00	0.129020E+01
0.750000E+03	0.999989E+00	0.129021E+01
0.100000E+04	0.999992E+00	0.129021E+01
0.200000E+04	0.999996E+00	0.129022E+01
0.300000E+04	0.999997E+00	0.129022E+01
0.400000E+04	0.999998E+00	0.129022E+01
0.500000E+04	0.999998E+00	0.129023E+01
0.750000E+04	0.999999E+00	0.129023E+01
0.100000E+05	0.999999E+00	0.129023E+01

TABLE 6-8. (CONT.)

DELTAT/PERIOD	$\beta = 1.00$ SPECTRAL RADIUS BY HUGHES	$\beta = 1.00$ SPECTRAL NORM BY HUGHES	$\beta = 1.50$ SPECTRAL RADIUS BY HUGHES	$\beta = 1.50$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162831E+01	0.100000E+01	0.176997E+01
0.500000E-01	0.971804E+00	0.145024E+01	0.973000E+00	0.151736E+01
0.100000E+00	0.955923E+00	0.121049E+01	0.961286E+00	0.116687E+01
0.200000E+00	0.952408E+00	0.109685E+01	0.963367E+00	0.107874E+01
0.300000E+00	0.959423E+00	0.121426E+01	0.970650E+00	0.111825E+01
0.500000E+00	0.971503E+00	0.137539E+01	0.980316E+00	0.118584E+01
0.750000E+00	0.979896E+00	0.145170E+01	0.986358E+00	0.122388E+01
0.100000E+01	0.984596E+00	0.148331E+01	0.989620E+00	0.124099E+01
0.150000E+01	0.989562E+00	0.150860E+01	0.993004E+00	0.125574E+01
0.200000E+01	0.992123E+00	0.151849E+01	0.994731E+00	0.126202E+01
0.300000E+01	0.994724E+00	0.152637E+01	0.996476E+00	0.126749E+01
0.400000E+01	0.996035E+00	0.152952E+01	0.997354E+00	0.126990E+01
0.500000E+01	0.996825E+00	0.153116E+01	0.997882E+00	0.127123E+01
0.100000E+02	0.998410E+00	0.153385E+01	0.998940E+00	0.127367E+01
0.200000E+02	0.999205E+00	0.153490E+01	0.999470E+00	0.127476E+01
0.300000E+02	0.999470E+00	0.153520E+01	0.999646E+00	0.127511E+01
0.400000E+02	0.999602E+00	0.153534E+01	0.999735E+00	0.127528E+01
0.500000E+02	0.999682E+00	0.153543E+01	0.999788E+00	0.127538E+01
0.750000E+02	0.999788E+00	0.153554E+01	0.999859E+00	0.127551E+01
0.100000E+03	0.999841E+00	0.153559E+01	0.999894E+00	0.127558E+01
0.200000E+03	0.999920E+00	0.153567E+01	0.999947E+00	0.127567E+01
0.300000E+03	0.999947E+00	0.153569E+01	0.999965E+00	0.127571E+01
0.400000E+03	0.999960E+00	0.153571E+01	0.999973E+00	0.127572E+01
0.500000E+03	0.999968E+00	0.153571E+01	0.999979E+00	0.127573E+01
0.750000E+03	0.999979E+00	0.153572E+01	0.999986E+00	0.127575E+01
0.100000E+04	0.999984E+00	0.153573E+01	0.999989E+00	0.127575E+01
0.200000E+04	0.999992E+00	0.153574E+01	0.999995E+00	0.127576E+01
0.300000E+04	0.999995E+00	0.153574E+01	0.999996E+00	0.127577E+01
0.400000E+04	0.999996E+00	0.153574E+01	0.999997E+00	0.127577E+01
0.500000E+04	0.999997E+00	0.153574E+01	0.999998E+00	0.127577E+01
0.750000E+04	0.999998E+00	0.153574E+01	0.999999E+00	0.127577E+01
0.100000E+05	0.999998E+00	0.153574E+01	0.999999E+00	0.127577E+01

TABLE 6-9. SPECTRAL NORM AND RADIUS FOR HUGHES METHOD FOR α = 0.0, γ = 1.0, θ = 1.0, ξ = 0.10

	β = 0.50		β = 1.0	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161803E+01	0.100000E+01	0.167459E+01
0.500000E-01	0.948227E+00	0.147024E+01	0.950484E+00	0.145749E+01
0.100000E+00	0.869383E+00	0.129188E+01	0.887427E+00	0.117277E+01
0.200000E+00	0.699986E+00	0.152490E+01	0.795143E+00	0.111550E+01
0.300000E+00	0.563122E+00	0.203763E+01	0.750456E+00	0.130247E+01
0.500000E+00	0.390342E+00	0.268792E+01	0.718445E+00	0.152044E+01
0.750000E+00	0.458296E+00	0.304375E+01	0.707948E+00	0.162571E+01
0.100000E+01	0.682762E+00	0.320951E+01	0.704929E+00	0.167133E+01
0.150000E+01	0.831439E+00	0.335659E+01	0.703687E+00	0.170988E+01
0.200000E+01	[illegible]89034E+00	0.342047E+01	0.703785E+00	0.172599E+01
0.300000E+01	0.936369E+00	0.347654E+01	0.704382E+00	0.173975E+01
0.400000E+01	0.956195E+00	0.350135E+01	0.704871E+00	0.174570E+01
0.500000E+01	0.966849E+00	0.351514E+01	0.705226E+00	0.174896E+01
0.100000E+02	0.985334E+00	0.354013E+01	0.706074E+00	0.175478E+01
0.200000E+02	0.993149E+00	0.355130E+01	0.706567E+00	0.175732E+01
0.300000E+02	0.995540E+00	0.355482E+01	0.706742E+00	0.175812E+01
0.400000E+02	0.996696E+00	0.355654E+01	0.706831E+00	0.175851E+01
0.500000E+02	0.997376E+00	0.355756E+01	0.706885E+00	0.175874E+01
0.750000E+02	0.998268E+00	0.355891E+01	0.706958E+00	0.175904E+01
0.100000E+03	0.998707E+00	0.355958E+01	0.706995E+00	0.175919E+01
0.200000E+03	0.999359E+00	0.356057E+01	0.707051E+00	0.175941E+01
0.300000E+03	0.999573E+00	0.356090E+01	0.707069E+00	0.175948E+01
0.400000E+03	0.999680E+00	0.356106E+01	0.707079E+00	0.175952E+01
0.500000E+03	0.999745E+00	0.356116E+01	0.707084E+00	0.175954E+01
0.750000E+03	0.999830E+00	0.356129E+01	0.707092E+00	0.175957E+01
0.100000E+04	0.999872E+00	0.356136E+01	0.707096E+00	0.175958E+01
0.200000E+04	0.999936E+00	0.356145E+01	0.707101E+00	0.175961E+01
0.300000E+04	0.999958E+00	0.356149E+01	0.707103E+00	0.175961E+01
0.400000E+04	0.999968E+00	0.356150E+01	0.707104E+00	0.175962E+01
0.500000E+04	0.999975E+00	0.356151E+01	0.707105E+00	0.175962E+01
0.750000E+04	0.999983E+00	0.356153E+01	0.707105E+00	0.175962E+01
0.100000E+05	0.999987E+00	0.356153E+01	0.707106E+00	0.175962E+01

TABLE 6-9. (CONT.)

DELTAT/PERIOD	β = 1.50 SPECTRAL RADIUS BY HUGHES	β = 1.50 SPECTRAL NORM BY HUGHES	β = 2.0 SPECTRAL RADIUS BY HUGHES	β = 2.0 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.184776E+01	0.100000E+01	0.212909E+01
0.500000E-01	0.952553E+00	0.154916E+01	0.954455E+00	0.172812E+01
0.100000E+00	0.901078E+00	0.116726E+01	0.911769E+00	0.123909E+01
0.200000E+00	0.844075E+00	0.101640E+01	0.874041E+00	0.104702E+01
0.300000E+00	0.823950E+00	0.111562E+01	0.863794E+00	0.110793E+01
0.500000E+00	0.813302E+00	0.123372E+01	0.860027E+00	0.118567E+01
0.750000E+00	0.811356E+00	0.128989E+01	0.860317E+00	0.122367E+01
0.100000E+01	0.811456E+00	0.131410E+01	0.861088E+00	0.124035E+01
0.150000E+01	0.812332E+00	0.133457E+01	0.862296E+00	0.125472E+01
0.200000E+01	0.813070E+00	0.134317E+01	0.863064E+00	0.126089E+01
0.300000E+01	0.814010E+00	0.135058E+01	0.863943E+00	0.126632E+01
0.400000E+01	0.814556E+00	0.135382E+01	0.864423E+00	0.126873E+01
0.500000E+01	0.814908E+00	0.135560E+01	0.864724E+00	0.127009E+01
0.100000E+02	0.815666E+00	0.135882E+01	0.865355E+00	0.127257E+01
0.200000E+02	0.816072E+00	0.136025E+01	0.865686E+00	0.127370E+01
0.300000E+02	0.816212E+00	0.136070E+01	0.865798E+00	0.127406E+01
0.400000E+02	0.816282E+00	0.136092E+01	0.865854E+00	0.127424E+01
0.500000E+02	0.816325E+00	0.136105E+01	0.865888E+00	0.127434E+01
0.750000E+02	0.816382E+00	0.136122E+01	0.865934E+00	0.127448E+01
0.100000E+03	0.816410E+00	0.136131E+01	0.865957E+00	0.127455E+01
0.200000E+03	0.816453E+00	0.136144E+01	0.865991E+00	0.127465E+01
0.300000E+03	0.816468E+00	0.136148E+01	0.866002E+00	0.127469E+01
0.400000E+03	0.816475E+00	0.136150E+01	0.866008E+00	0.127470E+01
0.500000E+03	0.816479E+00	0.136151E+01	0.866012E+00	0.127471E+01
0.750000E+03	0.816485E+00	0.136153E+01	0.866016E+00	0.127473E+01
0.100000E+04	0.816488E+00	0.136154E+01	0.866019E+00	0.127473E+01
0.200000E+04	0.816492E+00	0.136155E+01	0.866022E+00	0.127474E+01
0.300000E+04	0.816494E+00	0.136155E+01	0.866023E+00	0.127475E+01
0.400000E+04	0.816494E+00	0.136156E+01	0.866024E+00	0.127475E+01
0.500000E+04	0.816495E+00	0.136156E+01	0.866024E+00	0.127475E+01
0.750000E+04	0.816495E+00	0.136156E+01	0.866024E+00	0.127475E+01
0.100000E+05	0.816496E+00	0.136156E+01	0.866025E+00	0.127475E+01

TABLE 6-10. SPECTRAL NORM AND RADIUS FOR HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.01111$, $\theta = 1.1$, $\xi = 0.10$

	$\beta = 0.398115$		$\beta = 0.4$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162594E+01	0.100000E+01	0.162580E+01
0.500000E-01	0.947943E+00	0.149819E+01	0.947954E+00	0.149781E+01
0.100000E+00	0.869438E+00	0.134596E+01	0.869541E+00	0.134513E+01
0.200000E+00	0.712344E+00	0.154846E+01	0.713070E+00	0.154553E+01
0.300000E+00	0.612235E+00	0.201709E+01	0.613826E+00	0.201120E+01
0.500000E+00	0.531664E+00	0.264248E+01	0.534342E+00	0.263200E+01
0.750000E+00	0.499617E+00	0.299809E+01	0.502905E+00	0.298458E+01
0.100000E+01	0.486723E+00	0.316777E+01	0.490316E+00	0.315271E+01
0.150000E+01	0.741544E+00	0.332134E+01	0.708767E+00	0.330481E+01
0.200000E+01	0.832626E+00	0.338933E+01	0.804674E+00	0.337211E+01
0.300000E+01	0.903274E+00	0.344997E+01	0.877199E+00	0.343214E+01
0.400000E+01	0.932666E+00	0.347721E+01	0.907102E+00	0.345909E+01
0.500000E+01	0.948579E+00	0.349250E+01	0.923239E+00	0.347421E+01
0.100000E+02	0.976691E+00	0.352057E+01	0.951669E+00	0.350198E+01
0.200000E+02	0.988935E+00	0.353330E+01	0.964020E+00	0.351458E+01
0.300000E+02	0.992752E+00	0.353735E+01	0.967867E+00	0.351858E+01
0.400000E+02	0.994611E+00	0.353934E+01	0.969739E+00	0.352055E+01
0.500000E+02	0.995711E+00	0.354052E+01	0.970847E+00	0.352172E+01
0.750000E+02	0.997159E+00	0.354207E+01	0.972305E+00	0.352326E+01
0.100000E+03	0.997875E+00	0.354285E+01	0.973026E+00	0.352402E+01
0.200000E+03	0.998939E+00	0.354400E+01	0.974097E+00	0.352516E+01
0.300000E+03	0.999291E+00	0.354438E+01	0.974451E+00	0.352554E+01
0.400000E+03	0.999466E+00	0.354457E+01	0.974628E+00	0.352573E+01
0.500000E+03	0.999572E+00	0.354469E+01	0.974734E+00	0.352584E+01
0.750000E+03	0.999712E+00	0.354484E+01	0.974875E+00	0.352599E+01
0.100000E+04	0.999782E+00	0.354492E+01	0.974945E+00	0.352607E+01
0.200000E+04	0.999886E+00	0.354503E+01	0.975051E+00	0.352618E+01
0.300000E+04	0.999921E+00	0.354507E+01	0.975086E+00	0.352622E+01
0.400000E+04	0.999939E+00	0.354509E+01	0.975104E+00	0.352624E+01
0.500000E+04	0.999949E+00	0.354510E+01	0.975114E+00	0.352625E+01
0.750000E+04	0.999963E+00	0.354511E+01	0.975128E+00	0.352626E+01
0.100000E+05	0.999970E+00	0.354512E+01	0.975135E+00	0.352627E+01

TABLE 6-10. (CONT.)

DELTAT/PERIOD	$\beta = 0.5$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.5$ SPECTRAL NORM BY HUGHES	$\beta = 0.6$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.6$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162008E+01	0.100000E+01	0.161804E+01
0.500000E-01	0.948517E+00	0.147980E+01	0.949068E+00	0.146569E+01
0.100000E+00	0.874770E+00	0.130479E+01	0.879609E+00	0.127125E+01
0.200000E+00	0.746805E+00	0.140979E+01	0.773250E+00	0.130572E+01
0.300000E+00	0.681635E+00	0.174647E+01	0.728144E+00	0.155507E+01
0.500000E+00	0.639864E+00	0.217725E+01	0.704435E+00	0.186737E+01
0.750000E+00	0.627472E+00	0.240998E+01	0.699721E+00	0.203047E+01
0.100000E+01	0.623819E+00	0.251770E+01	0.699292E+00	0.210444E+01
0.150000E+01	0.621934E+00	0.261317E+01	0.700126E+00	0.216906E+01
0.200000E+01	0.621669E+00	0.265474E+01	0.701006E+00	0.219687E+01
0.300000E+01	0.621863E+00	0.269139E+01	0.702189E+00	0.222119E+01
0.400000E+01	0.622134E+00	0.270770E+01	0.702894E+00	0.223193E+01
0.500000E+01	0.622353E+00	0.271680E+01	0.703352E+00	0.223790E+01
0.100000E+02	0.622919E+00	0.273340E+01	0.704348E+00	0.224873E+01
0.200000E+02	0.623266E+00	0.274088E+01	0.704886E+00	0.225358E+01
0.300000E+02	0.623391E+00	0.274324E+01	0.705071E+00	0.225511E+01
0.400000E+02	0.623455E+00	0.274440E+01	0.705165E+00	0.225586E+01
0.500000E+02	0.623494E+00	0.274509E+01	0.705221E+00	0.225630E+01
0.750000E+02	0.623547E+00	0.274600E+01	0.705297E+00	0.225689E+01
0.100000E+03	0.623574E+00	0.274645E+01	0.705335E+00	0.225718E+01
0.200000E+03	0.623615E+00	0.274712E+01	0.705393E+00	0.225761E+01
0.300000E+03	0.623628E+00	0.274734E+01	0.705412E+00	0.225775E+01
0.400000E+03	0.623635E+00	0.274745E+01	0.705421E+00	0.225783E+01
0.500000E+03	0.623639E+00	0.274752E+01	0.705427E+00	0.225787E+01
0.750000E+03	0.623645E+00	0.274761E+01	0.705435E+00	0.225793E+01
0.100000E+04	0.623647E+00	0.274765E+01	0.705439E+00	0.225795E+01
0.200000E+04	0.623651E+00	0.274772E+01	0.705444E+00	0.225800E+01
0.300000E+04	0.623653E+00	0.274774E+01	0.705446E+00	0.225801E+01
0.400000E+04	0.623654E+00	0.274775E+01	0.705447E+00	0.225802E+01
0.500000E+04	0.623654E+00	0.274776E+01	0.705448E+00	0.225802E+01
0.750000E+04	0.623654E+00	0.274777E+01	0.705449E+00	0.225803E+01
0.100000E+05	0.623655E+00	0.274777E+01	0.705449E+00	0.225803E+01

TABLE 6-10. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES ($\beta = 0.620658$)	SPECTRAL NORM BY HUGHES ($\beta = 0.620658$)
0.000000E+00	0.100000E+01	0.161808E+01
0.500000E-01	0.949181E+00	0.146324E+01
0.100000E+00	0.880563E+00	0.126506E+01
0.200000E+00	0.778023E+00	0.128740E+01
0.300000E+00	0.736036E+00	0.152245E+01
0.500000E+00	0.714876E+00	0.181612E+01
0.750000E+00	0.711158E+00	0.196863E+01
0.100000E+01	0.711124E+00	0.203756E+01
0.150000E+01	0.712273E+00	0.209764E+01
0.200000E+01	0.713280E+00	0.212344E+01
0.300000E+01	0.714568E+00	0.214597E+01
0.400000E+01	0.715316E+00	0.215591E+01
0.500000E+01	0.715797E+00	0.216143E+01
0.100000E+02	0.716834E+00	0.217144E+01
0.200000E+02	0.717389E+00	0.217591E+01
0.300000E+02	0.717579E+00	0.217732E+01
0.400000E+02	0.717675E+00	0.217801E+01
0.500000E+02	0.717733E+00	0.217842E+01
0.750000E+02	0.717811E+00	0.217896E+01
0.100000E+03	0.717850E+00	0.217923E+01
0.200000E+03	0.717909E+00	0.217963E+01
0.300000E+03	0.717928E+00	0.217976E+01
0.400000E+03	0.717938E+00	0.217983E+01
0.500000E+03	0.717944E+00	0.217987E+01
0.750000E+03	0.717952E+00	0.217992E+01
0.100000E+04	0.717956E+00	0.217995E+01
0.200000E+04	0.717962E+00	0.217998E+01
0.300000E+04	0.717964E+00	0.218000E+01
0.400000E+04	0.717965E+00	0.218000E+01
0.500000E+04	0.717965E+00	0.218001E+01
0.750000E+04	0.717966E+00	0.218001E+01
0.100000E+05	0.717967E+00	0.218002E+01

TABLE 6-11. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $a = 0.0$, $\gamma = 1.2$, $\theta = 1.1$, $\xi = 0.10$

	$\beta = 0.466306$		$\beta = 0.6$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161812E+01	0.100000E+01	0.162035E+01
0.500000E-01	0.940166E+00	0.147099E+01	0.940995E+00	0.145577E+01
0.100000E+00	0.843574E+00	0.130168E+01	0.851298E+00	0.125690E+01
0.200000E+00	0.646334E+00	0.149844E+01	0.695151E+00	0.134073E+01
0.300000E+00	0.532245E+00	0.192662E+01	0.621574E+00	0.163154E+01
0.500000E+00	0.457901E+00	0.247320E+01	0.580206E+00	0.198817E+01
0.750000E+00	0.430238E+00	0.277792E+01	0.568441E+00	0.217700E+01
0.100000E+01	0.586449E+00	0.292318E+01	0.564692E+00	0.226427E+01
0.150000E+01	0.782960E+00	0.305562E+01	0.562301E+00	0.234212E+01
0.200000E+01	0.853214E+00	0.311496E+01	0.561605E+00	0.237639E+01
0.300000E+01	0.912107E+00	0.316859E+01	0.561240E+00	0.240699E+01
0.400000E+01	0.937722E+00	0.319303E+01	0.561183E+00	0.242079E+01
0.500000E+01	0.951912E+00	0.320687E+01	0.561188E+00	0.242856E+01
0.100000E+02	0.977681E+00	0.323261E+01	0.561290E+00	0.244291E+01
0.200000E+02	0.989270E+00	0.324448E+01	0.561387E+00	0.244948E+01
0.300000E+02	0.992941E+00	0.324828E+01	0.561426E+00	0.245157E+01
0.400000E+02	0.994741E+00	0.325016E+01	0.561446E+00	0.245260E+01
0.500000E+02	0.995809E+00	0.325127E+01	0.561459E+00	0.245322E+01
0.750000E+02	0.997220E+00	0.325274E+01	0.561477E+00	0.245403E+01
0.100000E+03	0.997920E+00	0.325348E+01	0.561486E+00	0.245443E+01
0.200000E+03	0.998963E+00	0.325457E+01	0.561500E+00	0.245503E+01
0.300000E+03	0.999308E+00	0.325493E+01	0.561504E+00	0.245523E+01
0.400000E+03	0.999481E+00	0.325511E+01	0.561507E+00	0.245533E+01
0.500000E+03	0.999584E+00	0.325522E+01	0.561508E+00	0.245538E+01
0.750000E+03	0.999722E+00	0.325537E+01	0.561510E+00	0.245546E+01
0.100000E+04	0.999791E+00	0.325544E+01	0.561511E+00	0.245550E+01
0.200000E+04	0.999894E+00	0.325555E+01	0.561512E+00	0.245556E+01
0.300000E+04	0.999928E+00	0.325558E+01	0.561513E+00	0.245558E+01
0.400000E+04	0.999945E+00	0.325560E+01	0.561513E+00	0.245559E+01
0.500000E+04	0.999955E+00	0.325561E+01	0.561513E+00	0.245560E+01
0.750000E+04	0.999969E+00	0.325563E+01	0.561513E+00	0.245561E+01
0.100000E+05	0.999976E+00	0.325563E+01	0.561513E+00	0.245561E+01

TABLE 6-11. (CONT.)

DELTAT/PERIOD	β = 0.7 SPECTRAL RADIUS BY HUGHES	β = 0.7 SPECTRAL NORM BY HUGHES	β = 0.8 SPECTRAL RADIUS BY HUGHES	β = 0.8 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162633E+01	0.100000E+01	0.163597E+01
0.500000E-01	0.941600E+00	0.144875E+01	0.942194E+00	0.144521E+01
0.100000E+00	0.856605E+00	0.123070E+01	0.861554E+00	0.120963E+01
0.200000E+00	0.723264E+00	0.125377E+01	0.746435E+00	0.118655E+01
0.300000E+00	0.667684E+00	0.147699E+01	0.703342E+00	0.136172E+01
0.500000E+00	0.639060E+00	0.174705E+01	0.682970E+00	0.157286E+01
0.750000E+00	0.632287E+00	0.188640E+01	0.679125E+00	0.167977E+01
0.100000E+01	0.630703E+00	0.194978E+01	0.678719E+00	0.172782E+01
0.150000E+01	0.630245E+00	0.200566E+01	0.679253E+00	0.176981E+01
0.200000E+01	0.630421E+00	0.203003E+01	0.679856E+00	0.178799E+01
0.300000E+01	0.630860E+00	0.205164E+01	0.680676E+00	0.180401E+01
0.400000E+01	0.631177E+00	0.206132E+01	0.681166E+00	0.181116E+01
0.500000E+01	0.631399E+00	0.206676E+01	0.681485E+00	0.181516E+01
0.100000E+02	0.631911E+00	0.207675E+01	0.682178E+00	0.182249E+01
0.200000E+02	0.632202E+00	0.208130E+01	0.682553E+00	0.182581E+01
0.300000E+02	0.632304E+00	0.208275E+01	0.682682E+00	0.182686E+01
0.400000E+02	0.632356E+00	0.208346E+01	0.682748E+00	0.182738E+01
0.500000E+02	0.632388E+00	0.208388E+01	0.682787E+00	0.182769E+01
0.750000E+02	0.632430E+00	0.208444E+01	0.682840E+00	0.182810E+01
0.100000E+03	0.632451E+00	0.208472E+01	0.682867E+00	0.182830E+01
0.200000E+03	0.632484E+00	0.208513E+01	0.682907E+00	0.182860E+01
0.300000E+03	0.632494E+00	0.208527E+01	0.682920E+00	0.182870E+01
0.400000E+03	0.632500E+00	0.208533E+01	0.682927E+00	0.182875E+01
0.500000E+03	0.632503E+00	0.208537E+01	0.682931E+00	0.182878E+01
0.750000E+03	0.632507E+00	0.208543E+01	0.682936E+00	0.182882E+01
0.100000E+04	0.632510E+00	0.208546E+01	0.682939E+00	0.182884E+01
0.200000E+04	0.632513E+00	0.208550E+01	0.682943E+00	0.182887E+01
0.300000E+04	0.632514E+00	0.208551E+01	0.682944E+00	0.182888E+01
0.400000E+04	0.632514E+00	0.208552E+01	0.682945E+00	0.182888E+01
0.500000E+04	0.632515E+00	0.208552E+01	0.682945E+00	0.182888E+01
0.750000E+04	0.632515E+00	0.208553E+01	0.682946E+00	0.182889E+01
0.100000E+05	0.632515E+00	0.208553E+01	0.682946E+00	0.182889E+01

TABLE 6-11. (CONT.)

	β = 1.0		β = 1.2	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.166626E+01	0.100000E+01	0.171130E+01
0.500000E-01	0.943346E+00	0.144782E+01	0.944455E+00	0.146250E+01
0.100000E+00	0.870510E+00	0.117995E+01	0.878393E+00	0.116330E+01
0.200000E+00	0.782526E+00	0.109472E+01	0.809440E+00	0.104179E+01
0.300000E+00	0.755243E+00	0.121075E+01	0.791406E+00	0.112754E+01
0.500000E+00	0.744583E+00	0.135199E+01	0.785956E+00	0.123301E+01
0.750000E+00	0.743856E+00	0.142174E+01	0.786709E+00	0.128423E+01
0.100000E+01	0.744588E+00	0.145259E+01	0.787907E+00	0.130667E+01
0.150000E+01	0.746000E+00	0.147924E+01	0.789616E+00	0.132595E+01
0.200000E+01	0.746947E+00	0.149066E+01	0.790651E+00	0.133419E+01
0.300000E+01	0.748049E+00	0.150066E+01	0.791801E+00	0.134139E+01
0.400000E+01	0.748656E+00	0.150510E+01	0.792418E+00	0.134458E+01
0.500000E+01	0.749038E+00	0.150757E+01	0.792802E+00	0.134635E+01
0.100000E+02	0.749841E+00	0.151208E+01	0.793598E+00	0.134959E+01
0.200000E+02	0.750263E+00	0.151411E+01	0.794010E+00	0.135104E+01
0.300000E+02	0.750406E+00	0.151475E+01	0.794150E+00	0.135151E+01
0.400000E+02	0.750478E+00	0.151507E+01	0.794220E+00	0.135173E+01
0.500000E+02	0.750521E+00	0.151526E+01	0.794262E+00	0.135187E+01
0.750000E+02	0.750580E+00	0.151550E+01	0.794318E+00	0.135205E+01
0.100000E+03	0.750609E+00	0.151563E+01	0.794347E+00	0.135213E+01
0.200000E+03	0.750653E+00	0.151581E+01	0.794389E+00	0.135226E+01
0.300000E+03	0.750667E+00	0.151587E+01	0.794403E+00	0.135231E+01
0.400000E+03	0.750675E+00	0.151590E+01	0.794411E+00	0.135233E+01
0.500000E+03	0.750679E+00	0.151592E+01	0.794415E+00	0.135234E+01
0.750000E+03	0.750685E+00	0.151594E+01	0.794420E+00	0.135236E+01
0.100000E+04	0.750688E+00	0.151595E+01	0.794423E+00	0.135237E+01
0.200000E+04	0.750692E+00	0.151597E+01	0.794428E+00	0.135238E+01
0.300000E+04	0.750694E+00	0.151598E+01	0.794429E+00	0.135239E+01
0.400000E+04	0.750694E+00	0.151598E+01	0.794430E+00	0.135239E+01
0.500000E+04	0.750695E+00	0.151598E+01	0.794430E+00	0.135239E+01
0.750000E+04	0.750695E+00	0.151599E+01	0.794431E+00	0.135239E+01
0.100000E+05	0.750696E+00	0.151599E+01	0.794431E+00	0.135239E+01

TABLE 6-11. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES (β = 1.24309)	SPECTRAL NORM BY HUGHES (β = 1.24309)
0.000000E+00	0.100000E+01	0.172294E+01
0.500000E-01	0.944688E+00	0.146716E+01
0.100000E+00	0.879970E+00	0.116112E+01
0.200000E+00	0.814373E+00	0.103412E+01
0.300000E+00	0.797619E+00	0.111567E+01
0.500000E+00	0.793161E+00	0.121597E+01
0.750000E+00	0.794121E+00	0.126452E+01
0.100000E+01	0.795378E+00	0.128576E+01
0.150000E+01	0.797116E+00	0.130401E+01
0.200000E+01	0.798156E+00	0.131180E+01
0.300000E+01	0.799304E+00	0.131861E+01
0.400000E+01	0.799918E+00	0.132163E+01
0.500000E+01	0.800299E+00	0.132331E+01
0.100000E+02	0.801087E+00	0.132637E+01
0.200000E+02	0.801495E+00	0.132775E+01
0.300000E+02	0.801633E+00	0.132819E+01
0.400000E+02	0.801702E+00	0.132840E+01
0.500000E+02	0.801744E+00	0.132853E+01
0.750000E+02	0.801800E+00	0.132870E+01
0.100000E+03	0.801828E+00	0.132878E+01
0.200000E+03	0.801870E+00	0.132891E+01
0.300000E+03	0.801884E+00	0.132895E+01
0.400000E+03	0.801891E+00	0.132897E+01
0.500000E+03	0.801895E+00	0.132898E+01
0.750000E+03	0.801901E+00	0.132900E+01
0.100000E+04	0.801904E+00	0.132901E+01
0.200000E+04	0.801908E+00	0.132902E+01
0.300000E+04	0.801909E+00	0.132902E+01
0.400000E+04	0.801910E+00	0.132902E+01
0.500000E+04	0.801910E+00	0.132903E+01
0.750000E+04	0.801911E+00	0.132903E+01
0.100000E+05	0.801911E+00	0.132903E+01

TABLE 6-12. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.5$, $\theta = 1.1$, $\xi = 0.10$

	$\beta = 0.574609$		$\beta = 0.6$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.163222E+01	0.100000E+01	0.163493E+01
0.500000E-01	0.928175E+00	0.144783E+01	0.928357E+00	0.144674E+01
0.100000E+00	0.804128E+00	0.125187E+01	0.805879E+00	0.124431E+01
0.200000E+00	0.544976E+00	0.145624E+01	0.556944E+00	0.142586E+01
0.300000E+00	0.439350E+00	0.184389E+01	0.456536E+00	0.178900E+01
0.500000E+00	0.395527E+00	0.231085E+01	0.409524E+00	0.222324E+01
0.750000E+00	0.465845E+00	0.256569E+01	0.402934E+00	0.245815E+01
0.100000E+01	0.674298E+00	0.268740E+01	0.406264E+00	0.256973E+01
0.150000E+01	0.812124E+00	0.279957E+01	0.616939E+00	0.267219E+01
0.200000E+01	0.869051E+00	0.285063E+01	0.682807E+00	0.271868E+01
0.300000E+01	0.919222E+00	0.289756E+01	0.737272E+00	0.276133E+01
0.400000E+01	0.941856E+00	0.291930E+01	0.761106E+00	0.278105E+01
0.500000E+01	0.954657E+00	0.293176E+01	0.774422E+00	0.279233E+01
0.100000E+02	0.978506E+00	0.295526E+01	0.798945E+00	0.281361E+01
0.200000E+02	0.989548E+00	0.296629E+01	0.810184E+00	0.282357E+01
0.300000E+02	0.993098E+00	0.296986E+01	0.813781E+00	0.282680E+01
0.400000E+02	0.994848E+00	0.297162E+01	0.815553E+00	0.282839E+01
0.500000E+02	0.995890E+00	0.297267E+01	0.816607E+00	0.282933E+01
0.750000E+02	0.997270E+00	0.297406E+01	0.818001E+00	0.283059E+01
0.100000E+03	0.997957E+00	0.297475E+01	0.818695E+00	0.283122E+01
0.200000E+03	0.998981E+00	0.297579E+01	0.819729E+00	0.283215E+01
0.300000E+03	0.999321E+00	0.297613E+01	0.820072E+00	0.283246E+01
0.400000E+03	0.999491E+00	0.297630E+01	0.820243E+00	0.283262E+01
0.500000E+03	0.999593E+00	0.297641E+01	0.820346E+00	0.283271E+01
0.750000E+03	0.999728E+00	0.297654E+01	0.820483E+00	0.283283E+01
0.100000E+04	0.999796E+00	0.297661E+01	0.820551E+00	0.283289E+01
0.200000E+04	0.999898E+00	0.297671E+01	0.820654E+00	0.283299E+01
0.300000E+04	0.999932E+00	0.297675E+01	0.820688E+00	0.283302E+01
0.400000E+04	0.999949E+00	0.297677E+01	0.820705E+00	0.283303E+01
0.500000E+04	0.999959E+00	0.297678E+01	0.820715E+00	0.283304E+01
0.750000E+04	0.999972E+00	0.297679E+01	0.820729E+00	0.283306E+01
0.100000E+05	0.999979E+00	0.297680E+01	0.820736E+00	0.283306E+01

TABLE 6-12. (CONT.)

DELTAT/PERIOD	β = 0.7 SPECTRAL RADIUS BY HUGHES	β = 0.7 SPECTRAL NORM BY HUGHES	β = 0.8 SPECTRAL RADIUS BY HUGHES	β = 0.8 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.164774E+01	0.100000E+01	0.166391E+01
0.500000E-01	0.929062E+00	0.144469E+01	0.929754E+00	0.144592E+01
0.100000E+00	0.812489E+00	0.121873E+01	0.818672E+00	0.119898E+01
0.200000E+00	0.598076E+00	0.132303E+01	0.631767E+00	0.124273E+01
0.300000E+00	0.514649E+00	0.160677E+01	0.561520E+00	0.146818E+01
0.500000E+00	0.478874E+00	0.193902E+01	0.532001E+00	0.172931E+01
0.750000E+00	0.468137E+00	0.211363E+01	0.524513E+00	0.186359E+01
0.100000E+01	0.464046E+00	0.219510E+01	0.522091E+00	0.192538E+01
0.150000E+01	0.460718E+00	0.226895E+01	0.520423E+00	0.198080E+01
0.200000E+01	0.459332E+00	0.230211E+01	0.519851E+00	0.200547E+01
0.300000E+01	0.458131E+00	0.233230E+01	0.519446E+00	0.202778E+01
0.400000E+01	0.457600E+00	0.234617E+01	0.519306E+00	0.203797E+01
0.500000E+01	0.457304E+00	0.235407E+01	0.519241E+00	0.204376E+01
0.100000E+02	0.456764E+00	0.236892E+01	0.519155E+00	0.205460E+01
0.200000E+02	0.456520E+00	0.237584E+01	0.519135E+00	0.205962E+01
0.300000E+02	0.456442E+00	0.237808E+01	0.519131E+00	0.206124E+01
0.400000E+02	0.456404E+00	0.237918E+01	0.519130E+00	0.206204E+01
0.500000E+02	0.456382E+00	0.237983E+01	0.519129E+00	0.206251E+01
0.750000E+02	0.456352E+00	0.238070E+01	0.519129E+00	0.206314E+01
0.100000E+03	0.456337E+00	0.238114E+01	0.519128E+00	0.206346E+01
0.200000E+03	0.456315E+00	0.238178E+01	0.519128E+00	0.206392E+01
0.300000E+03	0.456308E+00	0.238200E+01	0.519128E+00	0.206408E+01
0.400000E+03	0.456304E+00	0.238210E+01	0.519128E+00	0.206416E+01
0.500000E+03	0.456302E+00	0.238217E+01	0.519128E+00	0.206420E+01
0.750000E+03	0.456299E+00	0.238225E+01	0.519128E+00	0.206426E+01
0.100000E+04	0.456298E+00	0.238230E+01	0.519128E+00	0.206430E+01
0.200000E+04	0.456295E+00	0.238236E+01	0.519128E+00	0.206434E+01
0.300000E+04	0.456295E+00	0.238238E+01	0.519128E+00	0.206436E+01
0.400000E+04	0.456294E+00	0.238239E+01	0.519128E+00	0.206436E+01
0.500000E+04	0.456294E+00	0.238240E+01	0.519128E+00	0.206437E+01
0.750000E+04	0.456294E+00	0.238241E+01	0.519128E+00	0.206438E+01
0.100000E+05	0.456294E+00	0.238241E+01	0.519128E+00	0.206438E+01

TABLE 6-12. (CONT.)

DELTAT/PERIOD	β = 1.0 SPECTRAL RADIUS BY HUGHES	β = 1.0 SPECTRAL NORM BY HUGHES	β = 1.5 SPECTRAL RADIUS BY HUGHES	β = 1.5 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.170605E+01	0.100000E+01	0.186630E+01
0.500000E-01	0.931100E+00	0.145713E+01	0.934254E+00	0.152771E+01
0.100000E+00	0.829913E+00	0.117294E+01	0.852796E+00	0.115783E+01
0.200000E+00	0.684005E+00	0.113179E+01	0.765707E+00	0.101651E+01
0.300000E+00	0.633084E+00	0.128184E+01	0.740685E+00	0.109518E+01
0.500000E+00	0.611223E+00	0.145553E+01	0.730147E+00	0.118669E+01
0.750000E+00	0.606608E+00	0.154223E+01	0.728534E+00	0.123062E+01
0.100000E+01	0.605565E+00	0.158131E+01	0.728594E+00	0.124990E+01
0.150000E+01	0.605250E+00	0.161582E+01	0.729199E+00	0.126659E+01
0.200000E+01	0.605344E+00	0.163096E+01	0.729697E+00	0.127379E+01
0.300000E+01	0.605596E+00	0.164451E+01	0.730321E+00	0.128014E+01
0.400000E+01	0.605780E+00	0.165065E+01	0.730680E+00	0.128299E+01
0.500000E+01	0.605908E+00	0.165411E+01	0.730909E+00	0.128458E+01
0.100000E+02	0.606205E+00	0.166055E+01	0.731402E+00	0.128752E+01
0.200000E+02	0.606374E+00	0.166351E+01	0.731664E+00	0.128886E+01
0.300000E+02	0.606433E+00	0.166446E+01	0.731754E+00	0.128929E+01
0.400000E+02	0.606463E+00	0.166493E+01	0.731800E+00	0.128950E+01
0.500000E+02	0.606482E+00	0.166521E+01	0.731827E+00	0.128963E+01
0.750000E+02	0.606506E+00	0.166558E+01	0.731864E+00	0.128979E+01
0.100000E+03	0.606519E+00	0.166576E+01	0.731882E+00	0.128987E+01
0.200000E+03	0.606537E+00	0.166603E+01	0.731910E+00	0.129000E+01
0.300000E+03	0.606544E+00	0.166612E+01	0.731919E+00	0.129004E+01
0.400000E+03	0.606547E+00	0.166617E+01	0.731924E+00	0.129006E+01
0.500000E+03	0.606549E+00	0.166619E+01	0.731927E+00	0.129007E+01
0.750000E+03	0.606551E+00	0.166623E+01	0.731930E+00	0.129008E+01
0.100000E+04	0.606552E+00	0.166625E+01	0.731932E+00	0.129009E+01
0.200000E+04	0.606554E+00	0.166628E+01	0.731935E+00	0.129010E+01
0.300000E+04	0.606555E+00	0.166628E+01	0.731936E+00	0.129011E+01
0.400000E+04	0.606555E+00	0.166629E+01	0.731936E+00	0.129011E+01
0.500000E+04	0.606555E+00	0.166629E+01	0.731937E+00	0.129011E+01
0.750000E+04	0.606556E+00	0.166630E+01	0.731937E+00	0.129011E+01
0.100000E+05	0.606556E+00	0.166630E+01	0.731937E+00	0.129011E+01

TABLE 6-12. (CONT.)

DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES ($\beta = 2.0$)	SPECTRAL NORM BY HUGHES ($\beta = 2.0$)
0.000000E+00	0.100000E+01	0.209988E+01
0.500000E-01	0.937137E+00	0.164724E+01
0.100000E+00	0.870290E+00	0.118460E+01
0.200000E+00	0.813427E+00	0.101290E+01
0.300000E+00	0.799871E+00	0.107198E+01
0.500000E+00	0.795059E+00	0.114048E+01
0.750000E+00	0.794951E+00	0.117254E+01
0.100000E+01	0.795464E+00	0.118649E+01
0.150000E+01	0.796344E+00	0.119854E+01
0.200000E+01	0.796918E+00	0.120373E+01
0.300000E+01	0.797579E+00	0.120833E+01
0.400000E+01	0.797941E+00	0.121039E+01
0.500000E+01	0.798169E+00	0.121155E+01
0.100000E+02	0.798647E+00	0.121370E+01
0.200000E+02	0.798898E+00	0.121468E+01
0.300000E+02	0.798983E+00	0.121499E+01
0.400000E+02	0.799026E+00	0.121515E+01
0.500000E+02	0.799052E+00	0.121524E+01
0.750000E+02	0.799086E+00	0.121536E+01
0.100000E+03	0.799104E+00	0.121542E+01
0.200000E+03	0.799130E+00	0.121551E+01
0.300000E+03	0.799138E+00	0.121554E+01
0.400000E+03	0.799143E+00	0.121555E+01
0.500000E+03	0.799145E+00	0.121556E+01
0.750000E+03	0.799149E+00	0.121557E+01
0.100000E+04	0.799151E+00	0.121558E+01
0.200000E+04	0.799153E+00	0.121559E+01
0.300000E+04	0.799154E+00	0.121559E+01
0.400000E+04	0.799154E+00	0.121559E+01
0.500000E+04	0.799155E+00	0.121559E+01
0.750000E+04	0.799155E+00	0.121560E+01
0.100000E+05	0.799155E+00	0.121560E+01

TABLE 6-13. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR α = 0.0, γ = 1.3, θ = 1.4, ξ = 0.1

	β = 0.7		β = 0.8	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161848E+01	0.100000E+01	0.161821E+01
0.500000E-01	0.941738E+00	0.146578E+01	0.942710E+00	0.145602E+01
0.100000E+00	0.878353E+00	0.128272E+01	0.885912E+00	0.126543E+01
0.200000E+00	0.834109E+00	0.117426E+01	0.856659E+00	0.113800E+01
0.300000E+00	0.827381E+00	0.120187E+01	0.856572E+00	0.114766E+01
0.500000E+00	0.826250E+00	0.125736E+01	0.860335E+00	0.118438E+01
0.750000E+00	0.826914E+00	0.128919E+01	0.863076E+00	0.120654E+01
0.100000E+01	0.827508E+00	0.130423E+01	0.864574E+00	0.121705E+01
0.150000E+01	0.828259E+00	0.131787E+01	0.866123E+00	0.122656E+01
0.200000E+01	0.828690E+00	0.132398E+01	0.866910E+00	0.123080E+01
0.300000E+01	0.829158E+00	0.132953E+01	0.867702E+00	0.123464E+01
0.400000E+01	0.829405E+00	0.133207E+01	0.868099E+00	0.123639E+01
0.500000E+01	0.829557E+00	0.133351E+01	0.868337E+00	0.123738E+01
0.100000E+02	0.829871E+00	0.133623E+01	0.868814E+00	0.123924E+01
0.200000E+02	0.830033E+00	0.133749E+01	0.869052E+00	0.124010E+01
0.300000E+02	0.830087E+00	0.133789E+01	0.869132E+00	0.124037E+01
0.400000E+02	0.830115E+00	0.133809E+01	0.869172E+00	0.124051E+01
0.500000E+02	0.830131E+00	0.133821E+01	0.869195E+00	0.124059E+01
0.750000E+02	0.830153E+00	0.133837E+01	0.869227E+00	0.124070E+01
0.100000E+03	0.830164E+00	0.133845E+01	0.869243E+00	0.124075E+01
0.200000E+03	0.830181E+00	0.133856E+01	0.869267E+00	0.124083E+01
0.300000E+03	0.830186E+00	0.133860E+01	0.869275E+00	0.124086E+01
0.400000E+03	0.830189E+00	0.133862E+01	0.869279E+00	0.124087E+01
0.500000E+03	0.830191E+00	0.133863E+01	0.869281E+00	0.124088E+01
0.750000E+03	0.830193E+00	0.133865E+01	0.869284E+00	0.124089E+01
0.100000E+04	0.830194E+00	0.133866E+01	0.869286E+00	0.124089E+01
0.200000E+04	0.830196E+00	0.133867E+01	0.869288E+00	0.124090E+01
0.300000E+04	0.830196E+00	0.133867E+01	0.869289E+00	0.124090E+01
0.400000E+04	0.830196E+00	0.133868E+01	0.869290E+00	0.124091E+01
0.500000E+04	0.830197E+00	0.133868E+01	0.869290E+00	0.124091E+01
0.750000E+04	0.830197E+00	0.133868E+01	0.869290E+00	0.124091E+01
0.100000E+05	0.830197E+00	0.133868E+01	0.869290E+00	0.124091E+01

TABLE 6-13. (CONT.)

DELTAT/PERIOD	$\beta = 0.9$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.9$ SPECTRAL NORM BY HUGHES	$\beta = 0.946915$ SPECTRAL RADIUS BY HUGHES	$\beta = 0.946915$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.162030E+01	0.100000E+01	0.162211E+01
0.500000E-01	0.943659E+00	0.144809E+01	0.944097E+00	0.144496E+01
0.100000E+00	0.892787E+00	0.125060E+01	0.895800E+00	0.124436E+01
0.200000E+00	0.874824E+00	0.110959E+01	0.882172E+00	0.109831E+01
0.300000E+00	0.878964E+00	0.110727E+01	0.887747E+00	0.109179E+01
0.500000E+00	0.885530E+00	0.113273E+01	0.895188E+00	0.111375E+01
0.750000E+00	0.889369E+00	0.114937E+01	0.899347E+00	0.112877E+01
0.100000E+01	0.891324E+00	0.115734E+01	0.901430E+00	0.113601E+01
0.150000E+01	0.893266E+00	0.116456E+01	0.903478E+00	0.114257E+01
0.200000E+01	0.894223E+00	0.116778E+01	0.904480E+00	0.114549E+01
0.300000E+01	0.895168E+00	0.117068E+01	0.905463E+00	0.114813E+01
0.400000E+01	0.895634E+00	0.117201E+01	0.905947E+00	0.114933E+01
0.500000E+01	0.895912E+00	0.117275E+01	0.906235E+00	0.115001E+01
0.100000E+02	0.896463E+00	0.117415E+01	0.906804E+00	0.115127E+01
0.200000E+02	0.896736E+00	0.117479E+01	0.907085E+00	0.115186E+01
0.300000E+02	0.896826E+00	0.117500E+01	0.907178E+00	0.115204E+01
0.400000E+02	0.896871E+00	0.117510E+01	0.907225E+00	0.115213E+01
0.500000E+02	0.896898E+00	0.117516E+01	0.907253E+00	0.115219E+01
0.750000E+02	0.896935E+00	0.117524E+01	0.907290E+00	0.115226E+01
0.100000E+03	0.896953E+00	0.117528E+01	0.907308E+00	0.115230E+01
0.200000E+03	0.896980E+00	0.117534E+01	0.907336E+00	0.115235E+01
0.300000E+03	0.896989E+00	0.117536E+01	0.907345E+00	0.115237E+01
0.400000E+03	0.896993E+00	0.117537E+01	0.907350E+00	0.115238E+01
0.500000E+03	0.896996E+00	0.117538E+01	0.907353E+00	0.115238E+01
0.750000E+03	0.896999E+00	0.117539E+01	0.907357E+00	0.115239E+01
0.100000E+04	0.897001E+00	0.117539E+01	0.907358E+00	0.115240E+01
0.200000E+04	0.897004E+00	0.117540E+01	0.907361E+00	0.115240E+01
0.300000E+04	0.897005E+00	0.117540E+01	0.907362E+00	0.115240E+01
0.400000E+04	0.897005E+00	0.117540E+01	0.907363E+00	0.115240E+01
0.500000E+04	0.897006E+00	0.117540E+01	0.907363E+00	0.115240E+01
0.750000E+04	0.897006E+00	0.117540E+01	0.907363E+00	0.115240E+01
0.100000E+05	0.897006E+00	0.117540E+01	0.907363E+00	0.115240E+01

TABLE 6-14. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = 0.0$, $\gamma = 1.9$, $\theta = 1.4$, $\xi = 0.1$

	$\beta = 1.18$		$\beta = 1.2$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.169001E+01	0.100000E+01	0.169356E+01
0.500000E-01	0.922943E+00	0.143312E+01	0.923148E+00	0.143342E+01
0.100000E+00	0.842503E+00	0.119876E+01	0.843917E+00	0.119723E+01
0.200000E+00	0.808539E+00	0.106871E+01	0.811616E+00	0.106527E+01
0.300000E+00	0.807411E+00	0.107270E+01	0.810986E+00	0.106822E+01
0.500000E+00	0.808878E+00	0.109937E+01	0.812800E+00	0.109409E+01
0.750000E+00	0.809980E+00	0.111578E+01	0.814055E+00	0.111010E+01
0.100000E+01	0.810559E+00	0.112363E+01	0.814705E+00	0.111776E+01
0.150000E+01	0.811136E+00	0.113082E+01	0.815348E+00	0.112476E+01
0.200000E+01	0.811419E+00	0.113407E+01	0.815663E+00	0.112792E+01
0.300000E+01	0.811698E+00	0.113705E+01	0.815972E+00	0.113082E+01
0.400000E+01	0.811835E+00	0.113843E+01	0.816123E+00	0.113216E+01
0.500000E+01	0.811916E+00	0.113922E+01	0.816213E+00	0.113292E+01
0.100000E+02	0.812076E+00	0.114071E+01	0.816391E+00	0.113436E+01
0.200000E+02	0.812156E+00	0.114140E+01	0.816478E+00	0.113504E+01
0.300000E+02	0.812182E+00	0.114163E+01	0.816507E+00	0.113526E+01
0.400000E+02	0.812195E+00	0.114174E+01	0.816522E+00	0.113537E+01
0.500000E+02	0.812203E+00	0.114181E+01	0.816530E+00	0.113543E+01
0.750000E+02	0.812213E+00	0.114190E+01	0.816542E+00	0.113552E+01
0.100000E+03	0.812219E+00	0.114194E+01	0.816548E+00	0.113556E+01
0.200000E+03	0.812226E+00	0.114200E+01	0.816556E+00	0.113562E+01
0.300000E+03	0.812229E+00	0.114203E+01	0.816559E+00	0.113564E+01
0.400000E+03	0.812230E+00	0.114204E+01	0.816561E+00	0.113565E+01
0.500000E+03	0.812231E+00	0.114204E+01	0.816562E+00	0.113566E+01
0.750000E+03	0.812232E+00	0.114205E+01	0.816563E+00	0.113567E+01
0.100000E+04	0.812233E+00	0.114206E+01	0.816563E+00	0.113567E+01
0.200000E+04	0.812234E+00	0.114206E+01	0.816564E+00	0.113568E+01
0.300000E+04	0.812234E+00	0.114207E+01	0.816564E+00	0.113568E+01
0.400000E+04	0.812234E+00	0.114207E+01	0.816565E+00	0.113568E+01
0.500000E+04	0.812234E+00	0.114207E+01	0.816565E+00	0.113568E+01
0.750000E+04	0.812234E+00	0.114207E+01	0.816565E+00	0.113568E+01
0.100000E+05	0.812234E+00	0.114207E+01	0.816565E+00	0.113568E+01

TABLE 6-14. (CONT.)

DELTAT/PERIOD	β = 1.5 SPECTRAL RADIUS BY HUGHES	β = 1.5 SPECTRAL NORM BY HUGHES	β = 2.0 SPECTRAL RADIUS BY HUGHES	β = 2.0 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.175626E+01	0.100000E+01	0.189911E+01
0.500000E-01	0.926126E+00	0.144305E+01	0.930699E+00	0.147698E+01
0.100000E+00	0.862737E+00	0.117922E+01	0.886439E+00	0.116331E+01
0.200000E+00	0.848847E+00	0.102683E+01	0.887743E+00	0.100033E+01
0.300000E+00	0.853012E+00	0.102389E+01	0.894553E+00	0.103669E+01
0.500000E+00	0.857918E+00	0.104869E+01	0.900739E+00	0.108291E+01
0.750000E+00	0.860476E+00	0.106378E+01	0.903741E+00	0.110249E+01
0.100000E+01	0.861718E+00	0.107077E+01	0.905169E+00	0.111077E+01
0.150000E+01	0.862912E+00	0.107699E+01	0.906532E+00	0.111787E+01
0.200000E+01	0.863488E+00	0.107975E+01	0.907186E+00	0.112094E+01
0.300000E+01	0.864046E+00	0.108223E+01	0.907819E+00	0.112367E+01
0.400000E+01	0.864319E+00	0.108336E+01	0.908128E+00	0.112490E+01
0.500000E+01	0.864480E+00	0.108400E+01	0.908310E+00	0.112560E+01
0.100000E+02	0.864797E+00	0.108519E+01	0.908669E+00	0.112690E+01
0.200000E+02	0.864953E+00	0.108575E+01	0.908846E+00	0.112751E+01
0.300000E+02	0.865005E+00	0.108593E+01	0.908904E+00	0.112770E+01
0.400000E+02	0.865030E+00	0.108602E+01	0.908933E+00	0.112780E+01
0.500000E+02	0.865046E+00	0.108607E+01	0.908951E+00	0.112785E+01
0.750000E+02	0.865066E+00	0.108614E+01	0.908974E+00	0.112793E+01
0.100000E+03	0.865076E+00	0.108617E+01	0.908986E+00	0.112797E+01
0.200000E+03	0.865092E+00	0.108622E+01	0.909003E+00	0.112802E+01
0.300000E+03	0.865097E+00	0.108624E+01	0.909009E+00	0.112804E+01
0.400000E+03	0.865099E+00	0.108625E+01	0.909012E+00	0.112805E+01
0.500000E+03	0.865101E+00	0.108625E+01	0.909013E+00	0.112805E+01
0.750000E+03	0.865103E+00	0.108626E+01	0.909016E+00	0.112806E+01
0.100000E+04	0.865104E+00	0.108626E+01	0.909017E+00	0.112807E+01
0.200000E+04	0.865106E+00	0.108627E+01	0.909019E+00	0.112807E+01
0.300000E+04	0.865106E+00	0.108627E+01	0.909019E+00	0.112807E+01
0.400000E+04	0.865106E+00	0.108627E+01	0.909019E+00	0.112807E+01
0.500000E+04	0.865106E+00	0.108627E+01	0.909020E+00	0.112807E+01
0.750000E+04	0.865107E+00	0.108627E+01	0.909020E+00	0.112808E+01
0.100000E+05	0.865107E+00	0.108627E+01	0.909020E+00	0.112808E+01

TABLE 6-14. (CONT.)

DELTAT/PERIOD	$\beta = 2.14991$ SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.195085E+01
0.500000E-01	0.931982E+00	0.149079E+01
0.100000E+00	0.892172E+00	0.116085E+01
0.200000E+00	0.896067E+00	0.100499E+01
0.300000E+00	0.903135E+00	0.105923E+01
0.500000E+00	0.909364E+00	0.110721E+01
0.750000E+00	0.912358E+00	0.112685E+01
0.100000E+01	0.913778E+00	0.113511E+01
0.150000E+01	0.915133E+00	0.114218E+01
0.200000E+01	0.915782E+00	0.114524E+01
0.300000E+01	0.916411E+00	0.114797E+01
0.400000E+01	0.916718E+00	0.114920E+01
0.500000E+01	0.916899E+00	0.114991E+01
0.100000E+02	0.917256E+00	0.115121E+01
0.200000E+02	0.917432E+00	0.115182E+01
0.300000E+02	0.917490E+00	0.115202E+01
0.400000E+02	0.917519E+00	0.115211E+01
0.500000E+02	0.917536E+00	0.115217E+01
0.750000E+02	0.917559E+00	0.115225E+01
0.100000E+03	0.917571E+00	0.115228E+01
0.200000E+03	0.917588E+00	0.115234E+01
0.300000E+03	0.917594E+00	0.115236E+01
0.400000E+03	0.917597E+00	0.115237E+01
0.500000E+03	0.917598E+00	0.115238E+01
0.750000E+03	0.917601E+00	0.115238E+01
0.100000E+04	0.917602E+00	0.115239E+01
0.200000E+04	0.917604E+00	0.115239E+01
0.300000E+04	0.917604E+00	0.115239E+01
0.400000E+04	0.917605E+00	0.115239E+01
0.500000E+04	0.917605E+00	0.115240E+01
0.750000E+04	0.917605E+00	0.115240E+01
0.100000E+05	0.917605E+00	0.115240E+01

TABLE 6-15. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = -0.10$, $\gamma = 1.6$, $\theta = 1.4$, $\xi = 0.10$

	$\beta = 0.576$		$\beta = 0.8$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.161803E+01	0.100000E+01	0.162393E+01
0.500000E-01	0.934064E+00	0.146917E+01	0.936150E+00	0.145111E+01
0.100000E+00	0.845997E+00	0.129734E+01	0.863501E+00	0.125594E+01
0.200000E+00	0.753720E+00	0.126241E+01	0.813459E+00	0.115052E+01
0.300000E+00	0.722226E+00	0.137253E+01	0.805193E+00	0.118807E+01
0.500000E+00	0.697854E+00	0.152789E+01	0.802195E+00	0.125560E+01
0.750000E+00	0.686215E+00	0.161750E+01	0.801608E+00	0.129438E+01
0.100000E+01	0.680683E+00	0.166159E+01	0.801491E+00	0.131300E+01
0.150000E+01	0.675441E+00	0.170333E+01	0.801477E+00	0.133022E+01
0.200000E+01	0.672958E+00	0.172285E+01	0.801505E+00	0.133811E+01
0.300000E+01	0.670584E+00	0.174120E+01	0.801557E+00	0.134540E+01
0.400000E+01	0.669442E+00	0.174989E+01	0.801591E+00	0.134880E+01
0.500000E+01	0.668773E+00	0.175493E+01	0.801615E+00	0.135076E+01
0.100000E+02	0.667471E+00	0.176461E+01	0.801667E+00	0.135447E+01
0.200000E+02	0.666839E+00	0.176923E+01	0.801697E+00	0.135623E+01
0.300000E+02	0.666631E+00	0.177074E+01	0.801707E+00	0.135680E+01
0.400000E+02	0.666528E+00	0.177149E+01	0.801712E+00	0.135708E+01
0.500000E+02	0.666466E+00	0.177193E+01	0.801716E+00	0.135725E+01
0.750000E+02	0.666384E+00	0.177253E+01	0.801720E+00	0.135747E+01
0.100000E+03	0.666343E+00	0.177282E+01	0.801722E+00	0.135758E+01
0.200000E+03	0.666282E+00	0.177326E+01	0.801725E+00	0.135775E+01
0.300000E+03	0.666261E+00	0.177341E+01	0.801726E+00	0.135781E+01
0.400000E+03	0.666251E+00	0.177348E+01	0.801727E+00	0.135783E+01
0.500000E+03	0.666245E+00	0.177353E+01	0.801727E+00	0.135785E+01
0.750000E+03	0.666237E+00	0.177359E+01	0.801728E+00	0.135787E+01
0.100000E+04	0.666233E+00	0.177362E+01	0.801728E+00	0.135788E+01
0.200000E+04	0.666227E+00	0.177366E+01	0.801728E+00	0.135790E+01
0.300000E+04	0.666225E+00	0.177368E+01	0.801728E+00	0.135790E+01
0.400000E+04	0.666224E+00	0.177368E+01	0.801728E+00	0.135791E+01
0.500000E+04	0.666223E+00	0.177369E+01	0.801728E+00	0.135791E+01
0.750000E+04	0.666222E+00	0.177369E+01	0.801728E+00	0.135791E+01
0.100000E+05	0.666222E+00	0.177370E+01	0.801728E+00	0.135791E+01

TABLE 6-15. (CONT.)

DELTAT/PERIOD	$\beta = 1.0$ SPECTRAL RADIUS BY HUGHES	$\beta = 1.0$ SPECTRAL NORM BY HUGHES	$\beta = 1.1$ SPECTRAL RADIUS BY HUGHES	$\beta = 1.1$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.163901E+01	0.100000E+01	0.164998E+01
0.500000E-01	0.937924E+00	0.144269E+01	0.938781E+00	0.144085E+01
0.100000E+00	0.876471E+00	0.123004E+01	0.882190E+00	0.121983E+01
0.200000E+00	0.848183E+00	0.109203E+01	0.861562E+00	0.107107E+01
0.300000E+00	0.843621E+00	0.110229E+01	0.864499E+00	0.107411E+01
0.500000E+00	0.852151E+00	0.114148E+01	0.869672E+00	0.110707E+01
0.750000E+00	0.854498E+00	0.116503E+01	0.872694E+00	0.112737E+01
0.100000E+01	0.855729E+00	0.117625E+01	0.874221E+00	0.113701E+01
0.150000E+01	0.856965E+00	0.118651E+01	0.875727E+00	0.114578E+01
0.200000E+01	0.857579E+00	0.119115E+01	0.876464E+00	0.114972E+01
0.300000E+01	0.858186E+00	0.119539E+01	0.877188E+00	0.115331E+01
0.400000E+01	0.858487E+00	0.119735E+01	0.877544E+00	0.115496E+01
0.500000E+01	0.858666E+00	0.119847E+01	0.877756E+00	0.115590E+01
0.100000E+02	0.859022E+00	0.120058E+01	0.878174E+00	0.115767E+01
0.200000E+02	0.859198E+00	0.120157E+01	0.878381E+00	0.115849E+01
0.300000E+02	0.859257E+00	0.120189E+01	0.878450E+00	0.115876E+01
0.400000E+02	0.859286E+00	0.120204E+01	0.878484E+00	0.115889E+01
0.500000E+02	0.859304E+00	0.120214E+01	0.878504E+00	0.115897E+01
0.750000E+02	0.859327E+00	0.120226E+01	0.878532E+00	0.115907E+01
0.100000E+03	0.859339E+00	0.120232E+01	0.878545E+00	0.115913E+01
0.200000E+03	0.859356E+00	0.120242E+01	0.878566E+00	0.115920E+01
0.300000E+03	0.859362E+00	0.120245E+01	0.878572E+00	0.115923E+01
0.400000E+03	0.859365E+00	0.120246E+01	0.878576E+00	0.115924E+01
0.500000E+03	0.859367E+00	0.120247E+01	0.878578E+00	0.115925E+01
0.750000E+03	0.859369E+00	0.120249E+01	0.878581E+00	0.115926E+01
0.100000E+04	0.859370E+00	0.120249E+01	0.878582E+00	0.115927E+01
0.200000E+04	0.859372E+00	0.120250E+01	0.878584E+00	0.115927E+01
0.300000E+04	0.859373E+00	0.120250E+01	0.878585E+00	0.115928E+01
0.400000E+04	0.859373E+00	0.120251E+01	0.878585E+00	0.115928E+01
0.500000E+04	0.859373E+00	0.120251E+01	0.878585E+00	0.115928E+01
0.750000E+04	0.859373E+00	0.120251E+01	0.878586E+00	0.115928E+01
0.100000E+05	0.859373E+00	0.120251E+01	0.878586E+00	0.115928E+01

TABLE 6-15. (CONT.)

DELTAT/PERIOD	β = 1.2 SPECTRAL RADIUS BY HUGHES	β = 1.2 SPECTRAL NORM BY HUGHES	β = 1.2213 SPECTRAL RADIUS BY HUGHES	β = 1.2213 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.166321E+01	0.100000E+01	0.166632E+01
0.500000E-01	0.939618E+00	0.144046E+01	0.939794E+00	0.144056E+01
0.100000E+00	0.887471E+00	0.121105E+01	0.888544E+00	0.120934E+01
0.200000E+00	0.873025E+00	0.105389E+01	0.875257E+00	0.105063E+01
0.300000E+00	0.877742E+00	0.105265E+01	0.880281E+00	0.104882E+01
0.500000E+00	0.883984E+00	0.108294E+01	0.886698E+00	0.107895E+01
0.750000E+00	0.887417E+00	0.110174E+01	0.890195E+00	0.109762E+01
0.100000E+01	0.889120E+00	0.111061E+01	0.891924E+00	0.110640E+01
0.150000E+01	0.890782E+00	0.111862E+01	0.893609E+00	0.111432E+01
0.200000E+01	0.891590E+00	0.112220E+01	0.894428E+00	0.111785E+01
0.300000E+01	0.892380E+00	0.112545E+01	0.895227E+00	0.112106E+01
0.400000E+01	0.892768E+00	0.112693E+01	0.895619E+00	0.112252E+01
0.500000E+01	0.892997E+00	0.112778E+01	0.895852E+00	0.112336E+01
0.100000E+02	0.893451E+00	0.112937E+01	0.896310E+00	0.112492E+01
0.200000E+02	0.893675E+00	0.113011E+01	0.896536E+00	0.112565E+01
0.300000E+02	0.893749E+00	0.113035E+01	0.896611E+00	0.112588E+01
0.400000E+02	0.893786E+00	0.113046E+01	0.896649E+00	0.112600E+01
0.500000E+02	0.893808E+00	0.113053E+01	0.896671E+00	0.112607E+01
0.750000E+02	0.893837E+00	0.113063E+01	0.896701E+00	0.112616E+01
0.100000E+03	0.893852E+00	0.113067E+01	0.896716E+00	0.112621E+01
0.200000E+03	0.893874E+00	0.113074E+01	0.896738E+00	0.112627E+01
0.300000E+03	0.893881E+00	0.113076E+01	0.896745E+00	0.112630E+01
0.400000E+03	0.893885E+00	0.113078E+01	0.896749E+00	0.112631E+01
0.500000E+03	0.893887E+00	0.113078E+01	0.896751E+00	0.112631E+01
0.750000E+03	0.893890E+00	0.113079E+01	0.896754E+00	0.112632E+01
0.100000E+04	0.893892E+00	0.113080E+01	0.896756E+00	0.112633E+01
0.200000E+04	0.893894E+00	0.113080E+01	0.896758E+00	0.112633E+01
0.300000E+04	0.893895E+00	0.113080E+01	0.896759E+00	0.112634E+01
0.400000E+04	0.893895E+00	0.113081E+01	0.896759E+00	0.112634E+01
0.500000E+04	0.893895E+00	0.113081E+01	0.896759E+00	0.112634E+01
0.750000E+04	0.893896E+00	0.113081E+01	0.896760E+00	0.112634E+01
0.100000E+05	0.893896E+00	0.113081E+01	0.896760E+00	0.112634E+01

TABLE 6-16. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = -0.10$, $\gamma = 2.0$, $\theta = 1.4$, $\xi = 0.10$

	$\beta = 0.784$		$\beta = 0.8$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.164343E+01	0.100000E+01	0.164518E+01
0.500000E-01	0.920002E+00	0.144330E+01	0.920165E+00	0.144284E+01
0.100000E+00	0.812453E+00	0.124624E+01	0.813856E+00	0.124381E+01
0.200000E+00	0.732656E+00	0.119219E+01	0.736719E+00	0.118481E+01
0.300000E+00	0.713878E+00	0.126631E+01	0.719207E+00	0.125447E+01
0.500000E+00	0.700330E+00	0.137019E+01	0.706854E+00	0.135317E+01
0.750000E+00	0.693800E+00	0.142933E+01	0.700965E+00	0.140932E+01
0.100000E+01	0.690602E+00	0.145841E+01	0.698090E+00	0.143687E+01
0.150000E+01	0.687466E+00	0.148607E+01	0.695275E+00	0.146305E+01
0.200000E+01	0.685930E+00	0.149911E+01	0.693896E+00	0.147536E+01
0.300000E+01	0.684418E+00	0.151147E+01	0.692539E+00	0.148703E+01
0.400000E+01	0.683673E+00	0.151737E+01	0.691870E+00	0.149259E+01
0.500000E+01	0.683229E+00	0.152081E+01	0.691473E+00	0.149582E+01
0.100000E+02	0.682351E+00	0.152745E+01	0.690685E+00	0.150208E+01
0.200000E+02	0.681917E+00	0.153065E+01	0.690295E+00	0.150509E+01
0.300000E+02	0.681773E+00	0.153170E+01	0.690166E+00	0.150607E+01
0.400000E+02	0.681701E+00	0.153222E+01	0.690101E+00	0.150656E+01
0.500000E+02	0.681658E+00	0.153253E+01	0.690062E+00	0.150685E+01
0.750000E+02	0.681601E+00	0.153294E+01	0.690011E+00	0.150724E+01
0.100000E+03	0.681572E+00	0.153315E+01	0.689985E+00	0.150744E+01
0.200000E+03	0.681529E+00	0.153345E+01	0.689947E+00	0.150773E+01
0.300000E+03	0.681515E+00	0.153356E+01	0.689934E+00	0.150782E+01
0.400000E+03	0.681508E+00	0.153361E+01	0.689927E+00	0.150787E+01
0.500000E+03	0.681504E+00	0.153364E+01	0.689924E+00	0.150790E+01
0.750000E+03	0.681498E+00	0.153368E+01	0.689918E+00	0.150794E+01
0.100000E+04	0.681495E+00	0.153370E+01	0.689916E+00	0.150796E+01
0.200000E+04	0.681491E+00	0.153373E+01	0.689912E+00	0.150799E+01
0.300000E+04	0.681489E+00	0.153374E+01	0.689911E+00	0.150800E+01
0.400000E+04	0.681489E+00	0.153375E+01	0.689910E+00	0.150800E+01
0.500000E+04	0.681488E+00	0.153375E+01	0.689910E+00	0.150800E+01
0.750000E+04	0.681488E+00	0.153375E+01	0.689909E+00	0.150801E+01
0.100000E+05	0.681487E+00	0.153376E+01	0.689909E+00	0.150801E+01

TABLE 6-16. (CONT.)

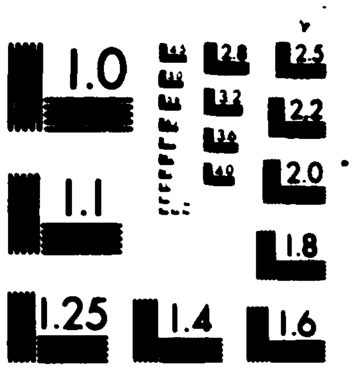
	β = 1.0		β = 1.5	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.167151E+01	0.100000E+01	0.177156E+01
0.500000E-01	0.922161E+00	0.144045E+01	0.926802E+00	0.145655E+01
0.100000E+00	0.829905E+00	0.121895E+01	0.860632E+00	0.118470E+01
0.200000E+00	0.778933E+00	0 111351E+01	0.843035E+00	0.102748E+01
0.300000E+00	0.772420E+00	0.114474E+01	0.846507E+00	0.103196E+01
0.500000E+00	0.769728E+00	0.120096E+01	0.851080E+00	0.106503E+01
0.750000E+00	0.768798E+00	0.123320E+01	0.853511E+00	0.108408E+01
0.100000E+01	0.768377E+00	0.124876E+01	0.854691E+00	0.109287E+01
0.150000E+01	0.767967E+00	0.126328E+01	0.855822E+00	0.110075E+01
0.200000E+01	0.767764E+00	0.126999E+01	0.856364E+00	0.110425E+01
0.300000E+01	0.767561E+00	0.127625E+01	0.856888E+00	0.110743E+01
0.400000E+01	0.767459E+00	0.127920E+01	0.857143E+00	0.110888E+01
0.500000E+01	0.767397E+00	0.128090E+01	0.857294E+00	0.110971E+01
0.100000E+02	0.767274E+00	0.128416E+01	0.857589E+00	0.111126E+01
0.200000E+02	0.767212E+00	0.128572E+01	0.857734E+00	0.111198E+01
0.300000E+02	0.767192E+00	0.128623E+01	0.857782E+00	0.111221E+01
0.400000E+02	0.767181E+00	0.128648E+01	0.857806E+00	0.111233E+01
0.500000E+02	0.767175E+00	0.128663E+01	0.857820E+00	0.111240E+01
0.750000E+02	0.767167E+00	0.128683E+01	0.857839E+00	0.111249E+01
0.100000E+03	0.767163E+00	0.128692E+01	0.857848E+00	0.111253E+01
0.200000E+03	0.767157E+00	0.128707E+01	0.857863E+00	0.111260E+01
0.300000E+03	0.767155E+00	0.128712E+01	0.857867E+00	0.111262E+01
0.400000E+03	0.767154E+00	0.128715E+01	0.857870E+00	0.111263E+01
0.500000E+03	0.767153E+00	0.128716E+01	0.857871E+00	0.111264E+01
0.750000E+03	0.767152E+00	0.128718E+01	0.857873E+00	0.111265E+01
0.100000E+04	0.767152E+00	0.128719E+01	0.857874E+00	0.111265E+01
0.200000E+04	0.767151E+00	0.128721E+01	0.857875E+00	0.111266E+01
0.300000E+04	0.767151E+00	0.128721E+01	0.857876E+00	0.111266E+01
0.400000E+04	0.767151E+00	0.128721E+01	0.857876E+00	0.111266E+01
0.500000E+04	0.767151E+00	0.128721E+01	0.857876E+00	0.111266E+01
0.750000E+04	0.767151E+00	0.128722E+01	0.857876E+00	0.111266E+01
0.100000E+05	0.767151E+00	0.128722E+01	0.857877E+00	0.111267E+01

TABLE 6-16. (CONT.)

DELTAT/PERIOD	$\beta = 2.0$		$\beta = 2.1104$	
	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.191758E+01	0.100000E+01	0.195568E+01
0.500000E-01	0.930991E+00	0.149606E+01	0.931861E+00	0.150732E+01
0.100000E+00	0.882421E+00	0.117104E+01	0.886394E+00	0.116971E+01
0.200000E+00	0.879012E+00	0.100017E+01	0.884915E+00	0.100111E+01
0.300000E+00	0.884946E+00	0.103418E+01	0.891053E+00	0.104655E+01
0.500000E+00	0.890742E+00	0.108322E+01	0.896892E+00	0.109758E+01
0.750000E+00	0.893626E+00	0.110487E+01	0.899778E+00	0.111935E+01
0.100000E+01	0.895009E+00	0.111413E+01	0.901161E+00	0.112860E+01
0.150000E+01	0.896335E+00	0.112212E+01	0.902436E+00	0.113656E+01
0.200000E+01	0.896972E+00	0.112559E+01	0.903123E+00	0.114001E+01
0.300000E+01	0.897590E+00	0.112868E+01	0.903741E+00	0.114309E+01
0.400000E+01	0.897891E+00	0.113008E+01	0.904043E+00	0.114449E+01
0.500000E+01	0.898070E+00	0.113088E+01	0.904222E+00	0.114528E+01
0.100000E+02	0.898421E+00	0.113236E+01	0.904573E+00	0.114676E+01
0.200000E+02	0.898593E+00	0.113304E+01	0.904746E+00	0.114745E+01
0.300000E+02	0.898650E+00	0.113326E+01	0.904804E+00	0.114767E+01
0.400000E+02	0.898679E+00	0.113337E+01	0.904832E+00	0.114777E+01
0.500000E+02	0.898696E+00	0.113344E+01	0.904849E+00	0.114784E+01
0.750000E+02	0.898719E+00	0.113352E+01	0.904872E+00	0.114792E+01
0.100000E+03	0.898730E+00	0.113357E+01	0.904883E+00	0.114797E+01
0.200000E+03	0.898747E+00	0.113363E+01	0.904901E+00	0.114803E+01
0.300000E+03	0.898753E+00	0.113365E+01	0.904906E+00	0.114805E+01
0.400000E+03	0.898755E+00	0.113366E+01	0.904909E+00	0.114806E+01
0.500000E+03	0.898757E+00	0.113367E+01	0.904911E+00	0.114807E+01
0.750000E+03	0.898759E+00	0.113368E+01	0.904913E+00	0.114808E+01
0.100000E+04	0.898761E+00	0.113368E+01	0.904914E+00	0.114808E+01
0.200000E+04	0.898762E+00	0.113369E+01	0.904916E+00	0.114809E+01
0.300000E+04	0.898763E+00	0.113369E+01	0.904916E+00	0.114809E+01
0.400000E+04	0.898763E+00	0.113369E+01	0.904917E+00	0.114809E+01
0.500000E+04	0.898763E+00	0.113369E+01	0.904917E+00	0.114809E+01
0.750000E+04	0.898763E+00	0.113369E+01	0.904917E+00	0.114809E+01
0.100000E+05	0.898764E+00	0.113369E+01	0.904917E+00	0.114809E+01

AD-A182 683 STABILITY LIMITS OF MODIFIED HILBER-HUGHES ALGORITHM 2/2
FOR TIME INTEGRATION IN DYNAMIC ANALYSIS(U) NAVAL
SURFACE WEAPONS CENTER SILVER SPRING MD M MOUSSOUROS
UNCLASSIFIED AUG 86 NSWC/TR-86-324 F/G 12/1 NL

END
8-87
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

TABLE 6-17. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = -0.2$, $\gamma = 1.3$, $\theta = 1.1$, $\xi = 0.1$

	$\beta = 3.46667$		$\beta = 4.0$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.305001E+01	0.100000E+01	0.347377E+01
0.500000E-01	0.956794E+00	0.225646E+01	0.958327E+00	0.248843E+01
0.100000E+00	0.926631E+00	0.146469E+01	0.932294E+00	0.155872E+01
0.200000E+00	0.908402E+00	0.117463E+01	0.918627E+00	0.122339E+01
0.300000E+00	0.905191E+00	0.121451E+01	0.916800E+00	0.125320E+01
0.500000E+00	0.904938E+00	0.126800E+01	0.917235E+00	0.129971E+01
0.750000E+00	0.905759E+00	0.129422E+01	0.918195E+00	0.132302E+01
0.100000E+01	0.906425F+00	0.130596E+01	0.918871E+00	0.133354E+01
0.150000E+01	0.907263E+00	0.131636E+01	0.919678E+00	0.134290E+01
0.200000E+01	0.907747E+00	0.132098E+01	0.920130E+00	0.134707E+01
0.300000E+01	0.908274E+00	0.132516E+01	0.920615E+00	0.135086E+01
0.400000E+01	0.908554E+00	0.132708E+01	0.920869E+00	0.135261E+01
0.500000E+01	0.908727E+00	0.132817E+01	0.921026E+00	0.135361E+01
0.100000E+02	0.909084E+00	0.133023E+01	0.921347E+00	0.135549E+01
0.200000E+02	0.909268E+00	0.133120E+01	0.921513E+00	0.135637E+01
0.300000E+02	0.909330E+00	0.133151E+01	0.921568E+00	0.135666E+01
0.400000E+02	0.909362E+00	0.133166E+01	0.921596E+00	0.135680E+01
0.500000E+02	0.909380E+00	0.133176E+01	0.921613E+00	0.135688E+01
0.750000E+02	0.909406E+00	0.133188E+01	0.921635E+00	0.135699E+01
0.100000E+03	0.909418E+00	0.133194E+01	0.921647E+00	0.135705E+01
0.200000E+03	0.909437E+00	0.133203E+01	0.921664E+00	0.135713E+01
0.300000E+03	0.909444E+00	0.133206E+01	0.921669E+00	0.135716E+01
0.400000E+03	0.909447E+00	0.133207E+01	0.921672E+00	0.135717E+01
0.500000E+03	0.909449E+00	0.133208E+01	0.921674E+00	0.135718E+01
0.750000E+03	0.909451E+00	0.133209E+01	0.921676E+00	0.135719E+01
0.100000E+04	0.909452E+00	0.133210E+01	0.921677E+00	0.135720E+01
0.200000E+04	0.909454E+00	0.133211E+01	0.921679E+00	0.135721E+01
0.300000E+04	0.909455E+00	0.133211E+01	0.921679E+00	0.135721E+01
0.400000E+04	0.909455E+00	0.133211E+01	0.921680E+00	0.135721E+01
0.500000E+04	0.909455E+00	0.133211E+01	0.921680E+00	0.135721E+01
0.750000E+04	0.909456E+00	0.133212E+01	0.921680E+00	0.135721E+01
0.100000E+05	0.909456E+00	0.133212E+01	0.921680E+00	0.135721E+01

TABLE 6-17. (CONT.)

DELTAT/PERIOD	$\beta = 5.0$ SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.430640E+01
0.500000E-01	0.960928E+00	0.290926E+01
0.100000E+00	0.940859E+00	0.171652E+01
0.200000E+00	0.932727E+00	0.129750E+01
0.300000E+00	0.932358E+00	0.131174E+01
0.500000E+00	0.933417E+00	0.134850E+01
0.750000E+00	0.934449E+00	0.136792E+01
0.100000E+01	0.935092E+00	0.137681E+01
0.150000E+01	0.935821E+00	0.138479E+01
0.200000E+01	0.936217E+00	0.138837E+01
0.300000E+01	0.936634E+00	0.139163E+01
0.400000E+01	0.936850E+00	0.139314E+01
0.500000E+01	0.936983E+00	0.139401E+01
0.100000E+02	0.937252E+00	0.139564E+01
0.200000E+02	0.937390E+00	0.139641E+01
0.300000E+02	0.937436E+00	0.139666E+01
0.400000E+02	0.937460E+00	0.139679E+01
0.500000E+02	0.937474E+00	0.139686E+01
0.750000E+02	0.937492E+00	0.139696E+01
0.100000E+03	0.937501E+00	0.139700E+01
0.200000E+03	0.937515E+00	0.139708E+01
0.300000E+03	0.937520E+00	0.139710E+01
0.400000E+03	0.937522E+00	0.139711E+01
0.500000E+03	0.937524E+00	0.139712E+01
0.750000E+03	0.937526E+00	0.139713E+01
0.100000E+04	0.937527E+00	0.139713E+01
0.200000E+04	0.937528E+00	0.139714E+01
0.300000E+04	0.937529E+00	0.139714E+01
0.400000E+04	0.937529E+00	0.139715E+01
0.500000E+04	0.937529E+00	0.139715E+01
0.750000E+04	0.937529E+00	0.139715E+01
0.100000E+05	0.937529E+00	0.139715E+01

TABLE 6-18. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = -0.2$, $\gamma = 1.5$, $\theta = 1.1$, $\xi = 0.1$

	$\beta = 1.14121$		$\beta = 1.5$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.174347E+01	0.100000E+01	0.186630E+01
0.500000E-01	0.940334E+00	0.149165E+01	0.942044E+00	0.155489E+01
0.100000E+00	0.858513E+00	0.118355E+01	0.870082E+00	0.118506E+01
0.200000E+00	0.743557E+00	0.109391E+01	0.781730E+00	0.102515E+01
0.300000E+00	0.697616E+00	0.126341E+01	0.747307E+00	0.113976E+01
0.500000E+00	0.672060E+00	0.146606E+01	0.725995E+00	0.127676E+01
0.750000E+00	0.665107E+00	0.156896E+01	0.719377E+00	0.134401E+01
0.100000E+01	0.663144E+00	0.161587E+01	0.717338E+00	0.137407E+01
0.150000E+01	0.662163E+00	0.165768E+01	0.716217E+00	0.140048E+01
0.200000E+01	0.662033E+00	0.167619E+01	0.716014E+00	0.141204E+01
0.300000E+01	0.662141E+00	0.169287E+01	0.716054E+00	0.142238E+01
0.400000E+01	0.662283E+00	0.170047E+01	0.716166E+00	0.142705E+01
0.500000E+01	0.662396E+00	0.170479E+01	0.716263E+00	0.142970E+01
0.100000E+02	0.662685E+00	0.171284E+01	0.716525E+00	0.143461E+01
0.200000E+02	0.662861E+00	0.171657E+01	0.716689E+00	0.143688E+01
0.300000E+02	0.662925E+00	0.171777E+01	0.716749E+00	0.143761E+01
0.400000E+02	0.662957E+00	0.171836E+01	0.716779E+00	0.143796E+01
0.500000E+02	0.662977E+00	0.171871E+01	0.716798E+00	0.143818E+01
0.750000E+02	0.663004E+00	0.171918E+01	0.716824E+00	0.143846E+01
0.100000E+03	0.663017E+00	0.171941E+01	0.716836E+00	0.143860E+01
0.200000E+03	0.663038E+00	0.171976E+01	0.716856E+00	0.143881E+01
0.300000E+03	0.663045E+00	0.171987E+01	0.716862E+00	0.143888E+01
0.400000E+03	0.663048E+00	0.171993E+01	0.716866E+00	0.143891E+01
0.500000E+03	0.663050E+00	0.171996E+01	0.716868E+00	0.143893E+01
0.750000E+03	0.663053E+00	0.172001E+01	0.716870E+00	0.143896E+01
0.100000E+04	0.663054E+00	0.172003E+01	0.716872E+00	0.143897E+01
0.200000E+04	0.663056E+00	0.172007E+01	0.716874E+00	0.143899E+01
0.300000E+04	0.663057E+00	0.172008E+01	0.716874E+00	0.143900E+01
0.400000E+04	0.663057E+00	0.172008E+01	0.716875E+00	0.143900E+01
0.500000E+04	0.663058E+00	0.172009E+01	0.716875E+00	0.143901E+01
0.750000E+04	0.663058E+00	0.172009E+01	0.716875E+00	0.143901E+01
0.100000E+05	0.663058E+00	0.172009E+01	0.716875E+00	0.143901E+01

TABLE 6-18. (CONT.)

DELTAT/PERIOD	$\beta = 2.0$ SPECTRAL RADIUS BY HUGHES	$\beta = 2.0$ SPECTRAL NORM BY HUGHES	$\beta = 3.0$ SPECTRAL RADIUS BY HUGHES	$\beta = 3.0$ SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.209988E+01	0.100000E+01	0.273543E+01
0.500000E-01	0.944270E+00	0.168768E+01	0.948246E+00	0.205667E+01
0.100000E+00	0.883398E+00	0.122248E+01	0.903264E+00	0.136685E+01
0.200000E+00	0.820637E+00	0.101565E+01	0.868520E+00	0.109846E+01
0.300000E+00	0.799119E+00	0.109916E+01	0.859392E+00	0.115355E+01
0.500000E+00	0.786650E+00	0.119751E+01	0.855317E+00	0.122081E+01
0.750000E+00	0.783151E+00	0.124460E+01	0.854767E+00	0.125297E+01
0.100000E+01	0.782254E+00	0.126541E+01	0.854915E+00	0.126720E+01
0.150000E+01	0.781974E+00	0.128358E+01	0.855359E+00	0.127970E+01
0.200000E+01	0.782074E+00	0.129151E+01	0.855694E+00	0.128519E+01
0.300000E+01	0.782339E+00	0.129859E+01	0.856105E+00	0.129013E+01
0.400000E+01	0.782533E+00	0.130179E+01	0.856340E+00	0.129239E+01
0.500000E+01	0.782670E+00	0.130360E+01	0.856489E+00	0.129367E+01
0.100000E+02	0.782988E+00	0.130697E+01	0.856809E+00	0.129606E+01
0.200000E+02	0.783170E+00	0.130852E+01	0.856979E+00	0.129717E+01
0.300000E+02	0.783234E+00	0.130901E+01	0.857038E+00	0.129753E+01
0.400000E+02	0.783267E+00	0.130926E+01	0.857067E+00	0.129771E+01
0.500000E+02	0.783286E+00	0.130941E+01	0.857085E+00	0.129781E+01
0.750000E+02	0.783313E+00	0.130960E+01	0.857109E+00	0.129795E+01
0.100000E+03	0.783326E+00	0.130969E+01	0.857121E+00	0.129802E+01
0.200000E+03	0.783347E+00	0.130984E+01	0.857139E+00	0.129813E+01
0.300000E+03	0.783353E+00	0.130989E+01	0.857145E+00	0.129816E+01
0.400000E+03	0.783357E+00	0.130991E+01	0.857148E+00	0.129818E+01
0.500000E+03	0.783359E+00	0.130992E+01	0.857149E+00	0.129819E+01
0.750000E+03	0.783362E+00	0.130994E+01	0.857152E+00	0.129820E+01
0.100000E+04	0.783363E+00	0.130995E+01	0.857153E+00	0.129821E+01
0.200000E+04	0.783365E+00	0.130997E+01	0.857155E+00	0.129822E+01
0.300000E+04	0.783366E+00	0.130997E+01	0.857155E+00	0.129822E+01
0.400000E+04	0.783366E+00	0.130997E+01	0.857156E+00	0.129823E+01
0.500000E+04	0.783366E+00	0.130997E+01	0.857156E+00	0.129823E+01
0.750000E+04	0.783366E+00	0.130998E+01	0.857156E+00	0.129823E+01
0.100000E+05	0.783367E+00	0.130998E+01	0.857156E+00	0.129823E+01

TABLE 6-18. (CONT.)

DELTAT/PERIOD	β = 4.0 SPECTRAL RADIUS BY HUGHES	β = 4.0 SPECTRAL NORM BY HUGHES	β = 6.0 SPECTRAL RADIUS BY HUGHES	β = 6.0 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.350080E+01	0.100000E+01	0.432670E+01
0.500000E-01	0.951692E+00	0.247579E+01	0.954707E+00	0.288727E+01
0.100000E+00	0.917359E+00	0.153548E+01	0.927873E+00	0.168899E+01
0.200000E+00	0.896385E+00	0.119207E+01	0.914534E+00	0.126742E+01
0.300000E+00	0.892149E+00	0.122708E+01	0.912575E+00	0.128666E+01
0.500000E+00	0.890989E+00	0.127857E+01	0.912612E+00	0.132773E+01
0.750000E+00	0.891327E+00	0.130395E+01	0.913250E+00	0.134890E+01
0.100000E+01	0.891744E+00	0.131531E+01	0.913731E+00	0.135850E+01
0.150000E+01	0.892331E+00	0.132536E+01	0.914322E+00	0.136707E+01
0.200000E+01	0.892688E+00	0.132981E+01	0.914658E+00	0.137089E+01
0.300000E+01	0.893089E+00	0.133385E+01	0.915022E+00	0.137436E+01
0.400000E+01	0.893305E+00	0.133570E+01	0.915214E+00	0.137596E+01
0.500000E+01	0.893440E+00	0.133675E+01	0.915333E+00	0.137688E+01
0.100000E+02	0.893721E+00	0.133874E+01	0.915577E+00	0.137860E+01
0.200000E+02	0.893867E+00	0.133967E+01	0.915703E+00	0.137941E+01
0.300000E+02	0.893917E+00	0.133997E+01	0.915745E+00	0.137967E+01
0.400000E+02	0.893941E+00	0.134011E+01	0.915767E+00	0.137980E+01
0.500000E+02	0.893957E+00	0.134020E+01	0.915780E+00	0.137988E+01
0.750000E+02	0.893977E+00	0.134032E+01	0.915797E+00	0.137998E+01
0.100000E+03	0.893987E+00	0.134038E+01	0.915805E+00	0.138003E+01
0.200000E+03	0.894002E+00	0.134046E+01	0.915818E+00	0.138011E+01
0.300000E+03	0.894007E+00	0.134049E+01	0.915823E+00	0.138014E+01
0.400000E+03	0.894009E+00	0.134051E+01	0.915825E+00	0.138015E+01
0.500000E+03	0.894011E+00	0.134052E+01	0.915826E+00	0.138016E+01
0.750000E+03	0.894013E+00	0.134053E+01	0.915828E+00	0.138017E+01
0.100000E+04	0.894014E+00	0.134053E+01	0.915829E+00	0.138017E+01
0.200000E+04	0.894015E+00	0.134054E+01	0.915830E+00	0.138018E+01
0.300000E+04	0.894016E+00	0.134054E+01	0.915830E+00	0.138018E+01
0.400000E+04	0.894016E+00	0.134055E+01	0.915830E+00	0.138018E+01
0.500000E+04	0.894016E+00	0.134055E+01	0.915831E+00	0.138018E+01
0.750000E+04	0.894017E+00	0.134055E+01	0.915831E+00	0.138018E+01
0.100000E+05	0.894017E+00	0.134055E+01	0.915831E+00	0.138018E+01

TABLE 6-19. SPECTRAL NORM AND RADIUS BY HUGHES METHOD FOR $\alpha = -0.2$, $\gamma = 2.0$, $\theta = 1.1$, $\xi = 0.1$

	$\beta = 0.862687$		$\beta = 0.9$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.176811E+01	0.100000E+01	0.177921E+01
0.500000E-01	0.918010E+00	0.149118E+01	0.918268E+00	0.149625E+01
0.100000E+00	0.771430E+00	0.119548E+01	0.773826E+00	0.119169E+01
0.200000E+00	0.584356E+00	0.136542E+01	0.585412E+00	0.133462E+01
0.300000E+00	0.585286E+00	0.172067E+01	0.583305E+00	0.166982E+01
0.500000E+00	0.601402E+00	0.213508E+01	0.597254E+00	0.205794E+01
0.750000E+00	0.611370E+00	0.236238E+01	0.606164E+00	0.226895E+01
0.100000E+01	0.616318E+00	0.247297E+01	0.610608E+00	0.237107E+01
0.150000E+01	0.621030E+00	0.257734E+01	0.614849E+00	0.246708E+01
0.200000E+01	0.623247E+00	0.262611E+01	0.616847E+00	0.251180E+01
0.300000E+01	0.625346E+00	0.267202E+01	0.618738E+00	0.255381E+01
0.400000E+01	0.626344E+00	0.269377E+01	0.619639E+00	0.257367E+01
0.500000E+01	0.626926E+00	0.270640E+01	0.620164E+00	0.258520E+01
0.100000E+02	0.628048E+00	0.273066E+01	0.621176E+00	0.260731E+01
0.200000E+02	0.628587E+00	0.274227E+01	0.621663E+00	0.261788E+01
0.300000E+02	0.628763E+00	0.274606E+01	0.621823E+00	0.262133E+01
0.400000E+02	0.628851E+00	0.274794E+01	0.621902E+00	0.262304E+01
0.500000E+02	0.628903E+00	0.274906E+01	0.621949E+00	0.262407E+01
0.750000E+02	0.628972E+00	0.275056E+01	0.622012E+00	0.262542E+01
0.100000E+03	0.629007E+00	0.275130E+01	0.622043E+00	0.262610E+01
0.200000E+03	0.629059E+00	0.275241E+01	0.622090E+00	0.262711E+01
0.300000E+03	0.629076E+00	0.275278E+01	0.622105E+00	0.262745E+01
0.400000E+03	0.629085E+00	0.275297E+01	0.622113E+00	0.262761E+01
0.500000E+03	0.629090E+00	0.275308E+01	0.622118E+00	0.262772E+01
0.750000E+03	0.629097E+00	0.275323E+01	0.622124E+00	0.262785E+01
0.100000E+04	0.629100E+00	0.275330E+01	0.622127E+00	0.262792E+01
0.200000E+04	0.629105E+00	0.275341E+01	0.622132E+00	0.262802E+01
0.300000E+04	0.629107E+00	0.275345E+01	0.622133E+00	0.262805E+01
0.400000E+04	0.629108E+00	0.275347E+01	0.622134E+00	0.262807E+01
0.500000E+04	0.629108E+00	0.275348E+01	0.622135E+00	0.262808E+01
0.750000E+04	0.629109E+00	0.275349E+01	0.622135E+00	0.262809E+01
0.100000E+05	0.629109E+00	0.275350E+01	0.622136E+00	0.262810E+01

TABLE 6-19. (CONT.)

DELTAT/PERIOD	β = 1.0 SPECTRAL RADIUS BY HUGHES	β = 1.0 SPECTRAL NORM BY HUGHES	β = 1.5 SPECTRAL RADIUS BY HUGHES	β = 1.5 SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.181050E+01	0.100000E+01	0.200000E+01
0.500000E-01	0.918950E+00	0.151120E+01	0.922198E+00	0.161093E+01
0.100000E+00	0.780012E+00	0.118474E+01	0.806490E+00	0.119110E+01
0.200000E+00	0.589784E+00	0.126216E+01	0.641739E+00	0.106291E+01
0.300000E+00	0.579531E+00	0.155102E+01	0.595032E+00	0.121915E+01
0.500000E+00	0.588103E+00	0.188026E+01	0.579702E+00	0.139558E+01
0.750000E+00	0.594522E+00	0.205554E+01	0.576972E+00	0.148355E+01
0.100000E+01	0.597808E+00	0.213928E+01	0.576400E+00	0.152375E+01
0.150000E+01	0.600978E+00	0.221727E+01	0.576231E+00	0.155995E+01
0.200000E+01	0.602481E+00	0.225333E+01	0.576275E+00	0.157620E+01
0.300000E+01	0.603912E+00	0.228701E+01	0.576394E+00	0.159105E+01
0.400000E+01	0.604596E+00	0.230286E+01	0.576479E+00	0.159791E+01
0.500000E+01	0.604995E+00	0.231204E+01	0.576539E+00	0.160184E+01
0.100000E+02	0.605767E+00	0.232960E+01	0.576675E+00	0.160926E+01
0.200000E+02	0.606139E+00	0.233797E+01	0.576751E+00	0.161274E+01
0.300000E+02	0.606261E+00	0.234069E+01	0.576778E+00	0.161387E+01
0.400000E+02	0.606322E+00	0.234204E+01	0.576792E+00	0.161443E+01
0.500000E+02	0.606358E+00	0.234285E+01	0.576800E+00	0.161476E+01
0.750000E+02	0.606406E+00	0.234392E+01	0.576811E+00	0.161520E+01
0.100000E+03	0.606430E+00	0.234446E+01	0.576817E+00	0.161542E+01
0.200000E+03	0.606466E+00	0.234525E+01	0.576825E+00	0.161575E+01
0.300000E+03	0.606478E+00	0.234552E+01	0.576828E+00	0.161586E+01
0.400000E+03	0.606484E+00	0.234565E+01	0.576829E+00	0.161591E+01
0.500000E+03	0.606488E+00	0.234573E+01	0.576830E+00	0.161594E+01
0.750000E+03	0.606493E+00	0.234584E+01	0.576831E+00	0.161599E+01
0.100000E+04	0.606495E+00	0.234589E+01	0.576832E+00	0.161601E+01
0.200000E+04	0.606499E+00	0.234597E+01	0.576833E+00	0.161604E+01
0.300000E+04	0.606500E+00	0.234600E+01	0.576833E+00	0.161605E+01
0.400000E+04	0.606500E+00	0.234601E+01	0.576833E+00	0.161606E+01
0.500000E+04	0.606501E+00	0.234602E+01	0.576833E+00	0.161606E+01
0.750000E+04	0.606501E+00	0.234603E+01	0.576833E+00	0.161606E+01
0.100000E+05	0.606501E+00	0.234603E+01	0.576833E+00	0.161607E+01

TABLE 6-19. (CONT.)

	$\beta = 2.0$	
DELTAT/PERIOD	SPECTRAL RADIUS BY HUGHES	SPECTRAL NORM BY HUGHES
0.000000E+00	0.100000E+01	0.224255E+01
0.500000E-01	0.925194E+00	0.174475E+01
0.100000E+00	0.827259E+00	0.123038E+01
0.200000E+00	0.706128E+00	0.101307E+01
0.300000E+00	0.659295E+00	0.111583E+01
0.500000E+00	0.631204E+00	0.123632E+01
0.750000E+00	0.622191E+00	0.129458E+01
0.100000E+01	0.619029E+00	0.132060E+01
0.150000E+01	0.616753E+00	0.134358E+01
0.200000E+01	0.615943E+00	0.135372E+01
0.300000E+01	0.615349E+00	0.136286E+01
0.400000E+01	0.615134E+00	0.136703E+01
0.500000E+01	0.615030E+00	0.136940E+01
0.100000E+02	0.614880E+00	0.137384E+01
0.200000E+02	0.614834E+00	0.137591E+01
0.300000E+02	0.614823E+00	0.137658E+01
0.400000E+02	0.614819E+00	0.137690E+01
0.500000E+02	0.614816E+00	0.137710E+01
0.750000E+02	0.614813E+00	0.137736E+01
0.100000E+03	0.614812E+00	0.137749E+01
0.200000E+03	0.614810E+00	0.137768E+01
0.300000E+03	0.614809E+00	0.137774E+01
0.400000E+03	0.614809E+00	0.137777E+01
0.500000E+03	0.614809E+00	0.137779E+01
0.750000E+03	0.614808E+00	0.137782E+01
0.100000E+04	0.614808E+00	0.137783E+01
0.200000E+04	0.614808E+00	0.137785E+01
0.300000E+04	0.614808E+00	0.137786E+01
0.400000E+04	0.614808E+00	0.137786E+01
0.500000E+04	0.614808E+00	0.137786E+01
0.750000E+04	0.614808E+00	0.137786E+01
0.100000E+05	0.614808E+00	0.137787E+01

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NOMENCLATURE

$||\underset{\sim}{A}||$ = Norm of amplification matrix

$\underset{\sim}{A}$ = Amplification matrix

a_{ij} = element (i,j) of amplification matrix $\underset{\sim}{A}$

B = Defined differently for Newmark β and Wilson θ methods

For Newmark β $$B = \frac{4\pi^2 p^2}{[1+4\pi\gamma\xi p+4\pi^2\beta p^2]}$$

For Wilson θ $$B = \frac{24\pi^2 p^2}{[6\theta+12\pi\xi\theta^2 p+4\pi^2\theta^3 p^2]}$$

For Hughes $$B = \frac{4\pi^2 p^2}{\theta[1+4\pi\gamma\theta\xi p+4\pi^2(1+\alpha)\beta\theta^2 p^2]}$$

c = Damping coefficient of SDOF (Lbf - sec/in)

$\underset{\sim}{C}$ = Damping matrix, usually taken as a linear combination of $\underset{\sim}{M}$ and $\underset{\sim}{K}$ matrices (Rayleigh damping) (Lbf - sec/in)

f = Decoupled force component (i.e., force of SDOF) (Lbf)

$\underset{\sim}{F}$, $\underset{\sim}{F}(t)$ = External force vector, dependent on time (Lbf)

k = Stiffness of SDOF (Lbf/in)

K = Stiffness matrix (symmetric and positive semidefinite (Lbf/in)

m = Mass of single degree of freedom system (SDOF) $\frac{\text{Lbf sec}^2}{\text{in}}$

NOMENCLATURE (Continued)

$\underset{\sim}{M}$ = Mass matrix (symmetric and positive definite, assuming that if there are degrees of freedom without mass, they have been condensed out)

$$\left(\frac{\text{Lbf sec}^2}{\text{in}}\right)$$

n = Integer indicating time step $n\Delta t$

p = $\frac{\Delta t}{T}$

T = Period of vibration (sec) ($= \frac{2\pi}{\omega}$)

Δt = Time step during integration process (sec)

$\underset{\sim}{X}_n$ = State vector defined by $X_n = \left[\Delta t^2 \ddot{x}_n,\ \Delta t \dot{x}_n,\ x_n\right]^T$

x_n, $\dot{x}_n$, $\ddot{x}_n$ = Displacement (in), velocity (in/sec), and acceleration of SDOF (in/sec^2)

α = Parameter in α-method

β = Parameter of approximating displacements in Newmark-β method and Hilber-Hughes methods

γ = Parameter approximating velocities in both Newmark β and Hilber-Hughes methods

ξ = Modal damping parameter for an uncoupled system ($2\xi\omega = \frac{c}{m}$)

θ = Parameter of approximating accelerations in Wilson θ and Hilber-Hughes methods. Also equilibrium equations are satisfied at point $n+\theta$.

κ = As defined by $\kappa = \frac{B\xi}{2\pi p}$

u_n, $\dot{u}_n$, $\ddot{u}_n$ = On Tables, displacement (in) (u_n), velocities (in/sec) ($\dot{u}_n$) and acceleration of SDOF (in/sec^2) ($\ddot{u}_n$)

ω = Frequency of vibration (rad/sec) of uncoupled system (SDOF)

($= \frac{2\pi}{T}$)

APPENDIX A

STABILITY ANALYSIS OF THE MODIFIED HILBER-HUGHES METHOD

In this section we perform an analysis of the stability characteristics of the modified α-method with collocation. To this end we consider the uncoupled equations of motion of a linear system. They will be satisfied at a collocation point $n+\theta$:

$$M\,\ddot{x}_{n+\theta} + C\,\dot{x}_{n+\theta} + (1+\alpha)K\,x_{n+\theta} - \alpha K\,x_n = F_{n+\theta} \tag{A-1}$$

where

$$\ddot{x}_{n+\theta} = (1-\theta)\ddot{x}_n + \theta\,\ddot{x}_{n+1} \tag{A-2}$$

$$\dot{x}_{n+\theta} = \dot{x}_n + \theta\Delta t[(1-\gamma)\ddot{x}_n + \gamma\ddot{x}_{n+\theta}] \tag{A-3}$$

$$x_{n+\theta} = x_n + \theta\Delta t\,\dot{x}_n + \theta(\theta\Delta t)^2\left[(\tfrac{1}{2}-\beta)\ddot{x}_n + \beta\ddot{x}_{n+\theta}\right] \tag{A-4}$$

Replace Equation (A-2) in Equation (A-3) to obtain the velocity

$$\dot{x}_{n+\theta} = \dot{x}_n + (1-\gamma\theta)\theta\Delta t\,\ddot{x}_n + \gamma\theta^2\Delta t\,\ddot{x}_{n+1} \tag{A-5}$$

Replace Equation (A-2) in Equation (A-4) to obtain

$$x_{n+\theta} = x_n + \theta\Delta t\,\dot{x}_n + (\tfrac{1}{2}-\beta\theta)\,\theta^2\Delta t^2\,\ddot{x}_n + \beta\theta^3\Delta t^2\,\ddot{x}_{n+1} \tag{A-6}$$

Substituting Equations (A-2), (A-5), and (A-6) for the acceleration, velocity, and displacement at the collocation point $n+\theta$ in terms of all the historical variables x_n, $\dot{x}_n$, $\ddot{x}_n$ and the acceleration at the next time step $n+1$, and making the substitutions

$$\omega^2 = \frac{K}{M} \tag{A-7}$$

$$2\omega\xi = \frac{C}{M} \tag{A-8}$$

and

$$[1+2\gamma\theta\xi\omega\Delta t+(1+\alpha)\beta\theta^2\omega^2\Delta t^2] = \frac{\omega^2\Delta t^2}{B} \tag{A-9}$$

$$\xi = \frac{\kappa\omega\Delta t}{B} \tag{A-10}$$

we obtain

$$\ddot{x}_{n+1} = \frac{B}{K\Delta t^2} F_{n+\theta} - \frac{B}{\Delta t^2} x_n - \frac{[2\kappa+(1+\alpha)\theta B]}{\Delta t}\dot{x}_n$$

$$- \left\{(1-\theta)\frac{B}{\omega^2\Delta t^2} + 2(1-\gamma\theta)\theta\kappa + (1+\alpha)(\tfrac{1}{2}-\beta\theta)\theta^2 B\right\}\ddot{x}_n \quad \text{(A-11)}$$

Replacing Equation (A-11) in the expression for velocity at time step n+1

$$\dot{x}_{n+1} = \dot{x}_n + \Delta t\left[(1-\gamma)\ddot{x}_n + \gamma\ \ddot{x}_{n+1}\right] \quad \text{(A-12)}$$

we obtain

$$\dot{x}_{n+1} = \frac{\gamma B}{K\Delta t} F_{n+\theta} - \frac{B\gamma}{\Delta t} x_n + \left[1-\left\{2\kappa+(1+\alpha)\theta B\right\}\gamma\right]\dot{x}_n$$

$$+ \Delta t\left[(1-\gamma) - \frac{(1-\theta)\gamma B}{\omega^2\Delta t^2} - 2(1-\gamma\theta)\gamma\theta\kappa + (1+\alpha)(\beta\theta-\tfrac{1}{2})\theta^2 B\gamma\right]\ddot{x}_n \quad \text{(A-13)}$$

The displacement is calculated from

$$x_{n+1} = x_n + \Delta t\ x_n + \Delta t^2\left[\left(\frac{1}{2} - \beta\right)\ddot{x}_n + \beta\ddot{x}_{n+\theta}\right]$$

$$= x_n + \Delta t\ \dot{x}_n + \left(\frac{1}{2} - \beta\right)\Delta t^2\ \ddot{x}_n + \beta\Delta t^2\ \ddot{x}_{n+1} \quad \text{(A-14)}$$

and Equation (A-11), i.e.

$$x_{n+1} = \frac{\beta B}{K} F_{n+\theta} + [1-\beta B]x_n + \left[1-\beta\left\{2\kappa+(1+\alpha)\theta B\right\}\right]\Delta t\ \dot{x}_n +$$

$$+ \left[\left(\tfrac{1}{2}-\beta\right) - \frac{(1-\theta)\beta B}{\omega^2\Delta t^2} - 2(1-\gamma\theta)\theta\beta\kappa + (1+\alpha)(\beta\theta-\tfrac{1}{2})\theta^2\beta B\right]\Delta t^2\ \ddot{x}_n \quad \text{(A-15)}$$

We may write Equations (A-11), (A-13), and (A-15) in matrix form

$$\begin{bmatrix} \Delta t^2 \ddot{x}_{n+1} \\ \Delta t \dot{x}_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \left[(1+\alpha)(\beta\theta - \frac{1}{2})\theta^2 B - 2(1-\gamma\theta)\theta\kappa - (1-\theta)\frac{B}{\omega^2 \Delta t^2}\right] & -[2\kappa+(1+\alpha)\theta B] & -B \\ \left[(1-\gamma) - \frac{(1-\theta)\gamma B}{\omega^2 \Delta t^2} - 2(1-\gamma\theta)\gamma\theta\kappa + (1+\alpha)(\beta\theta - \frac{1}{2})\theta^2 B\gamma\right] & 1-[2\kappa+(1+\alpha)\theta B]\gamma & -\gamma B \\ \left[(\frac{1}{2} - \beta) - \frac{(1-\theta)\beta B}{\omega^2 \Delta t^2} - 2(1-\gamma\theta)\theta\beta\kappa + (1+\alpha)(\beta\theta - \frac{1}{2})\theta^2 \beta B\right] & 1-\beta[2\kappa+(1+\alpha)\theta B] & [1-\beta B] \end{bmatrix} \begin{bmatrix} \Delta t^2 \ddot{x}_n \\ \Delta t \dot{x}_n \\ x_n \end{bmatrix} + \begin{bmatrix} \frac{B}{K} F_{n+\theta} \\ \frac{\gamma B}{K} F_{n+\theta} \\ \frac{\beta B}{K} F_{n+\theta} \end{bmatrix} \tag{A-16}$$

where

$$B = \frac{\omega^2 \Delta t^2}{\left[1+2\gamma\theta\xi\omega\Delta t+(1+\alpha)\beta\theta^2\omega^2\Delta t^2\right]} = \frac{4\pi^2 p^2}{\theta\left[1+4\pi\gamma\theta\xi p+4\pi^2(1+\alpha)\beta\theta^2 p^2\right]} \tag{A-17}$$

$$\xi = \frac{\kappa\omega\Delta t}{B} = \frac{2\pi\kappa p}{B} \tag{A-18}$$

$$p = \frac{\Delta t}{T} \tag{A-19}$$

or

$$\underset{\sim}{X}_{n+1} = \underset{\sim}{A}\, \underset{\sim}{X}_n + \underset{\sim}{L}\, \underset{\sim}{F}_{n+\theta}$$

We make the following substitutions in matrix A of Equation (A-16)

$$L_1 = (1+\alpha)(\beta\theta - \tfrac{1}{2})\theta^2 B - 2(1-\gamma\theta)\theta\kappa - (1-\theta)\frac{B}{\omega^2 \Delta t^2}$$

$$= \frac{1}{D}[(1+\alpha)(\beta\theta - \tfrac{1}{2})\theta^2\Omega^2 - 2(1-\gamma\theta)\theta\xi\Omega - (1-\theta)] \tag{A-21}$$

$$L_2 = -[2\kappa+(1+\alpha)\theta B] = -\frac{1}{D}[2\xi\Omega + (1+\alpha)\theta\Omega^2] \tag{A-22}$$

$$L_3 = -B = -\frac{1}{D}\Omega^2 \tag{A-23}$$

$$D = \theta\left[1+2\gamma\xi\theta\Omega + (1+\alpha)\beta\theta^2\Omega^2\right] \quad \text{(A-24)}$$

$$\xi = \frac{\kappa\omega\Delta t}{B} = \frac{\kappa\Omega}{B} \quad \text{(A-25)}$$

$$B = \frac{\Omega^2}{D} \quad \text{(A-26)}$$

The amplification matrix $\underset{\sim}{A}$ can be rewritten in the form

$$\underset{\sim}{A} = \begin{array}{|c|c|c|} \hline L_1 & L_2 & L_3 \\ \hline \left\{1+\gamma(L_1-1)\right\} & \left\{1+\gamma L_2\right\} & \gamma L_3 \\ \hline \left\{\frac{1}{2} + \beta(L_1-1)\right\} & \left\{1+\beta L_2\right\} & 1+\beta L_3 \\ \hline \end{array}$$

The characteristic equation stemming from $\underset{\sim}{A}$, is

$$\det|\underset{\sim}{A} - \lambda\underset{\sim}{I}| = -\lambda^3 + 2A_1\lambda^2 - A_2\lambda + A_3 = 0 \quad \text{(A-28)}$$

or

$$\lambda^3 - 2A_1\lambda^2 + A_2\lambda - A_3 = 0 \quad \text{(A-29)}$$

where the three invariants $2A_1$, A_2, A_3 are explicitly given as

$$2A_1 = 2 + L_1 + \gamma L_2 + \beta L_3 \quad \text{(A-30)}$$

$$A_2 = 1 + 2L_1 + (2\gamma-1)L_2 + (2\beta-\gamma-\tfrac{1}{2})L_3 \quad \text{(A-31)}$$

$$A_3 = L_1 - (1-\gamma)L_2 - (\gamma-\beta-\tfrac{1}{2})L_3 \quad \text{(A-32)}$$

or

$$2A_1 = 2 + \frac{1}{D}\Big[\left\{(1+\alpha)(\beta\theta - \tfrac{1}{2})\Omega^2 - (1+\alpha)\gamma\theta - \beta\right\}\Omega^2 + \left\{-2(1-\gamma\theta)\theta - 2\gamma\right\}\xi\Omega - (1-\theta)\Big] \quad \text{(A-33)}$$

$$A_2 = 1 + \frac{1}{D}\Big[\left\{2(1+\alpha)(\beta\theta-\tfrac{1}{2})\Omega^2 - (2\gamma-1)(1+\alpha)\theta - (2\beta-\gamma-\tfrac{1}{2})\right\}\Omega^2 + \left\{-4(1-\gamma\theta)\theta - 2(2\gamma-1)\right\}\xi\Omega - 2(1-\theta)\Big] \quad \text{(A-34)}$$

$$A_3 = \frac{1}{D}\left[\left\{(1+\alpha)(\beta\theta-\tfrac{1}{2})\theta^2+(1-\gamma)(1+\alpha)\theta+(\gamma-\beta-\tfrac{1}{2})\right\}\Omega^2 + \right.$$

$$\left. + \left\{-2(1-\gamma\theta)\,\theta+2(1-\gamma)\right\}\,\xi\Omega-(1-\theta)\right] \qquad \text{(A-35)}$$

and $\quad 2A_1 - A_3 = 2 + \frac{1}{D}\left[\Omega^2\left\{(\tfrac{1}{2}-\gamma)-(1+\alpha)\theta\right\}-2\xi\Omega\right]$

We distinguish two cases in our analysis depending on whether A_3 is zero.

(1) $A_3 \neq 0$ (full cubic equation)

In this case, for absolute stability we must satisfy the Routh-Hurwitz criterion

$$E_1 = 1 - 2A_1 + A_2 - A_3 = -L_3 > 0 \qquad \text{(A-36)}$$

$$E_2 = 3 - 2A_1 - A_2 + 3A_3 = -2L_2 + 2(1-\gamma)L_3 > 0 \qquad \text{(A-37)}$$

$$E_3 = 3 + 2A_1 - A_2 - 3A_3$$

$$= 4(1-L_1) + 4(1-\gamma)L_2 + (4\gamma-4\beta-1)L_3 > 0 \qquad \text{(A-38)}$$

$$E_4 = 1 + 2A_1 + A_2 + A_3$$

$$= 4(1+L_1) + 2(2\gamma-1)L_2 + 2L_3(2\beta-\gamma) > 0 \qquad \text{(A-39)}$$

$$E_5 = E_2E_3 - E_1E_4$$

$$= -8L_2 - 4(2\gamma-3)L_3 + 8L_1L_2 + 8(\gamma-1)L_2^2$$

$$+ 4L_2L_3\left\{2\gamma^2-5\gamma+2\beta+2\right\} + 2L_3^2\left\{4\gamma-2\beta+4\beta\gamma-4\gamma^2-1\right\}$$

$$- 4L_1L_3\left\{1-2\gamma\right\} > 0 \qquad \text{(A-40)}$$

or cast in terms of Ω

$$E_1 = \frac{\Omega^2}{D} > 0 \qquad \text{(A-41)}$$

$$D = \theta\left[1+2\gamma\xi\theta\Omega+(1+\alpha)\theta^2\Omega^2\right]$$

$$E_2 = \frac{1}{D}\left[2\left\{(1+\alpha)\theta+\gamma-1\right\}\Omega^2+4\xi\Omega\right] =$$

$$= \frac{2\Omega}{D}\left[\left\{(1+\alpha)\theta+\gamma-1\right\}\Omega+2\xi\right] \tag{A-42}$$

$$E_3 = \frac{1}{D}\left[\left\{2(1+\alpha)\theta^2+4(\gamma-1)(1+\alpha)\theta+(1+4\beta-4\gamma)\right\}\Omega^2+8\left\{\theta+(\gamma-1)\right\}\xi\Omega+4\right] \tag{A-43}$$

where the coefficient of Ω^2 in E_3 can be written as

$$[4\beta+ 2(1+\alpha)\theta^2 +4(1+\alpha)(\gamma-1)\theta+1-4\gamma]$$

$$E_4 = \frac{4}{D}\left\{\left\{(1+\alpha)(2\beta\theta^3-\frac{1}{2}\theta^2)+(1+\alpha)(1-2\gamma)\theta+(\frac{1}{2}\gamma-\beta)\right\}\Omega^2+\left\{1-2\gamma-2\theta+4\gamma\theta^2\right\}\xi\Omega+(2\theta-1)\right\}$$

$$= \frac{4}{D}\left\{\left\{[2(1+\alpha)\theta^3-1]\beta+\frac{1}{2}(1+\alpha)[(1-2\gamma)\theta-\theta^2]+\frac{\gamma}{2}\right\}\Omega^2+\left\{1-2\gamma-2\theta+4\gamma\theta^2\right\}\xi\Omega+(2\theta-1)\right\} \tag{A-44}$$

$$\begin{aligned} E_2E_3 - E_1E_4 = \frac{4\Omega}{D^2}\Bigg[&\Big\{\beta[-2(1+\alpha)\theta^3 + 2(1+\alpha)\theta + 2\gamma - 1] + (1+\alpha)^2\theta^2[\theta+2(\gamma-1)] \\ &+ (1+\alpha)\left[(\tfrac{1}{2}-2\gamma)\theta + (\gamma-1)\theta^2 + 2(\gamma-1)^2\theta - \tfrac{1}{2}\left\{(1-2\gamma)\theta - \theta^2\right\}\right] \\ &+ 2\gamma(1-\gamma) - \tfrac{1}{2}\Big\}\Omega^3 \\ &+ \left\{4\beta + (1+\alpha)(6\theta^2+8(\gamma-1)\theta) + 4(\gamma-1)\theta + 4(\gamma-1)^2 - 2\gamma + 2\theta - 4\gamma\theta^2\right\}\xi\Omega^2 \\ &+\left\{2\gamma(1+4\xi^2) + 2\alpha\theta - 1 + 8(\theta-1)\xi^2\right\}\Omega + 4\xi\Bigg] \end{aligned} \tag{A-45}$$

Therefore, for unconditional stability the following constraints must be applied to the coefficients of all the inequalities from Equations (A-41) through (A-45).

With the exception of the case when $\Omega = 0$ (yielding equality), Equation (A-41) is always satisfied for any positive Ω. Equation (A-42) leads to

$$\alpha \geq \frac{(1-\gamma)}{\theta} - 1 \tag{A-46}$$

or

$$\alpha \geq 1 - (1+\alpha)\theta \tag{A-47}$$

and

$$\xi \geq 0 \tag{A-48}$$

Equation (A-43) leads to

$$\beta \geq -\tfrac{1}{2}\left[2(1+\alpha)\theta^2 + 4(1+\alpha)(\gamma-1)\theta + (1-4\gamma)\right] \tag{A-49}$$

$$(\theta+\gamma-1)\xi \geq 0 \tag{A-50}$$

and, in view of Equation (A-48), Equation (A-50) becomes

$$\theta + \gamma - 1 \geq 0 \tag{A-51}$$

Equation (A-44) implies the following constraints ($\xi \geq 0$ has already been accounted for)

$$\gamma \geq \tfrac{1}{2} \tag{A-52}$$

To obtain Equation (A-52) we look for the roots of the equation

$$1 - 2\gamma - 2\theta + 4\gamma\theta^2 = 0$$

They are

$$\theta_{1,2} = \frac{1}{4\gamma} \pm \tfrac{1}{2}\sqrt{\left(\frac{1}{2\gamma} - 1\right)^2 + 1}$$

The smallest root of θ must be at least 1, i.e.

$$\frac{1}{4\gamma} + \tfrac{1}{2}\sqrt{\left(\frac{1}{2\gamma} - 1\right)^2 + 1} \geq 1$$

This gives Equation (A-52).

And if $\quad 2(1+\alpha)\theta^3 - 1 > 0$ (A-53)

then $\quad \beta \geq \dfrac{(1+\alpha)\theta[\theta-(1-2\gamma)]-\gamma}{2[2(1+\alpha)\theta^3-1]}$ (A-54)

However, if

$$2(1+\alpha)\theta^3 - 1 < 0 \tag{A-55}$$

and if

$$2(1+\alpha)\theta^3 - 1 = 0 \tag{A-57}$$

$$0 \geq (1+\alpha)\theta[\theta - (1-2\gamma)] - \gamma \tag{A-58}$$

Substituting the value for α from Equation (A-57) in Equation (A-58) we obtain

$$\gamma \geq \frac{1}{2(\theta+1)} \tag{A-59}$$

which is satisfied, since $\theta \geq 1$. And from Equation (A-52), we must have $\gamma \geq \frac{1}{2}$.

The same argument applies to Equation (A-51), which can be written as

$$\gamma \geq 1 - \theta.$$

γ must exceed a nonpositive number and it is superfluous.

Equation (A-45) leads to more complicated inequalities. They are

$$\gamma \geq \frac{1 - 2\alpha\theta - 8(\theta-1)\xi^2}{2(1+4\xi^2)} \tag{A-60}$$

where

$$1 - 2\alpha\theta - 8(\theta-1)\xi^2 \geq 0$$

or

$$(1+8\xi^2) \geq 2(\alpha+4\xi^2)\theta$$

and if $\quad \alpha + 4\xi^2 < 0$, then $\dfrac{(1+8\xi^2)}{2(\alpha+4\xi^2)} \leq \theta$

if $\quad \alpha + 4\xi^2 > 0$, then $\dfrac{1+8\xi^2}{2(\alpha+4\xi^2)} \geq \theta$,

and if $\alpha + 4\xi^2 = 0$, then no restraint on θ

$$\beta \geq \left[-\frac{1}{4}(1+\alpha)\left\{6\theta^2 + 8(\gamma-1)\theta\right\} + 4(\gamma-1)\theta + 4(\gamma-1)^2 + (2\theta-2\gamma-4\gamma\theta^2)\right] \tag{A-61}$$

Recall from algebra at this point that, given a polynomial of third degree

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

its three roots τ_1, τ_2, and τ_3 are given by

$$\tau_1 + \tau_2 + \tau_3 = -\frac{a_2}{a_3}$$

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = \frac{a_1}{a_3}$$

$$\tau_1\tau_2\tau_3 = -\frac{a_0}{a_3}$$

In our case, all the roots should be negative, because any nonnegative x would yield $f(x) \geq 0$. Therefore, we seek

$$\tau_1 + \tau_2 + \tau_3 \leq 0$$

$$\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 \geq 0$$

$$\tau_1\tau_2\tau_3 \leq 0$$

If $a_3 > 0$, then all $a_i \geq 0$ $(i = 0,1,2)$;

If $a_3 < 0$, then all $a_i \leq 0$ $(i = 0,1,2)$. But in this case any $x \geq 0$ will yield $f(x) \leq 0$ and this case does not satisfy the nonnegativity of $f(x)$. Hence it is of no interest to us.

If $a_3 = 0$, the polynomial is quadratic

$$f(x) = a_2x^2 + a_1x + a_0$$

and

$$\tau_1 + \tau_2 = -\frac{a_1}{a_2}$$

$$\tau_1\tau_2 = \frac{a_0}{a_2}$$

For τ_1, τ_2 to be nonpositive

$$\tau_1 + \tau_2 \leq 0$$

$$\tau_1\tau_2 \geq 0$$

or for

$$a_2 > 0 \qquad a_1, a_0 \geq 0$$

$$a_2 < 0 \qquad a_1, a_0 \leq 0$$

and again the case of all $a_i < 0$ does not yield $f(x) \geq 0$ for $x \geq 0$ and it is of no interest.

If $a_2 = 0$ (in addition to when $a_3 = 0$), then the polynomial is linear, i.e.

$$f(x) = a_1 x + a_0$$

and

$$\tau_1 = -\frac{a_0}{a_1}$$

For it to be nonpositive

$$\tau_1 \leq 0$$

and if $a_1 > 0 \qquad a_0 \geq 0$

if $a_1 < 0 \qquad a_0 < 0$

yielding $f(x) \leq 0$ for $x \geq 0$ and it is of no interest.

If $a_1 = 0$,

$$f(x) = a_0$$

and $a_0 \geq 0$

Consequently

$$\beta \geq - \frac{\left\{(1+\alpha)^2\left[\theta+2(\gamma-1)\right]\theta^2+(1+\alpha)\left[(\tfrac{1}{2}-2\gamma)\theta+(\gamma-1)\theta^2+2(\gamma-1)^2\theta+\tfrac{1}{2}\theta^2-(\tfrac{1}{2}-\gamma)\theta\right]+2\gamma(1-\gamma)-\tfrac{1}{2}\right\}}{\left\{2\gamma-1+2(1+\alpha)\theta-2(1+\alpha)\theta^3\right\}} \tag{A-62}$$

if $\qquad \gamma > \tfrac{1}{2} + (1+\alpha)\theta^3 - (1+\alpha)\theta$

However, the case of

$$\gamma < \tfrac{1}{2} + (1+\alpha)\theta^3 - (1+\alpha)\theta \tag{A-63}$$

cannot occur, since all the coefficients of Equation (A-45) must be negative, which is precluded due to $\xi \geq 0$.

If Equation (A-62) is not valid, then Equation (A-64) is the other possibility, i.e.

$$0 \geq - \left\{(1+\alpha)^2\left[\theta+2(\gamma-1)\right]\theta^2+(1+\alpha)\left[(\tfrac{1}{2}-2\gamma)\theta+(\gamma-1)\theta^2+2(\gamma-1)^2\theta+\tfrac{1}{2}\theta^2-(\tfrac{1}{2}-\gamma)\theta\right]+2\gamma(1-\gamma)-\tfrac{1}{2}\right\}$$

if $\qquad \gamma = \tfrac{1}{2} + (1+\alpha)\theta^3 - (1+\alpha)\theta^3 \qquad$ (A-64)

SUMMARY $(A_3 \neq 0)$

$$\gamma \geq 1 - (1+\alpha)\theta$$

$$\gamma \geq 1 - \theta$$

$$\gamma \geq \frac{1}{2}$$

$$\gamma \geq \tfrac{1}{2} + (1+\alpha)\theta(\theta^2-1)$$

$$\gamma \geq \frac{1 - 2\alpha\theta - 8(\theta-1)\xi^2}{2(1+4\xi^2)}$$

$$2(1+\alpha)\theta^3 \geq 1$$

$$\left(\text{if} \quad 2(1+\alpha)\theta^3 = 1 \quad \rightarrow \quad \gamma \geq \frac{1}{2(\theta+1)}\right)$$

$$\xi \geq 0$$

$$\beta \geq -\frac{\left\{(1+\alpha)^2\left[\theta+2(\gamma-1)\right]\theta^2+(1+\alpha)\ \left(\tfrac{1}{2}-2\gamma)\theta+(\gamma-1)\theta^2+2(\gamma-1)^2\theta+\tfrac{1}{2}\theta^2-(\tfrac{1}{2}-\gamma)\theta\right]+2\gamma(1-\gamma)-\tfrac{1}{2}\right\}}{\left\{2\gamma-1+2(1+\alpha)\theta-2(1+\alpha)\theta^3\right\}}$$

$$\beta \geq -\frac{1}{4}[2(1+\alpha)\theta^2+4(1+\alpha)(\gamma-1)\theta+(1-4\gamma)]$$

$$\beta \geq \frac{(1+\alpha)\theta[\theta-(1-2\gamma)] - \gamma}{2[2(1+\alpha)\theta^3-1]}$$

$$\beta \geq -\frac{1}{4}[(1+\alpha)\left\{6\theta^2+8(\gamma-1)\theta\right\}+4(\gamma-1)\theta+4(\gamma-1)^2+(2\theta-2\gamma-4\gamma\theta^2)]$$

A further point in the investigation is to examine whether it is possible to have complex roots of unit argument. This is given by the condition

$$1 - A_2 + A_3[2A_1-A_3] = 0 \tag{A-65}$$

which can be written in the form

$$a_4\Omega^4 + a_3\Omega^3 + a_2\Omega^2 + a_1\Omega =$$

$$= \Omega[a_4\Omega^3+a_3\Omega^2+a_2\Omega+a_1] = 0$$

where

$$a_1 = 2\xi \tag{A-67}$$

$$a_2 = (1+\alpha)\theta + \gamma - \theta - \tfrac{1}{2} + 4\xi^2(\theta-1+\gamma) \tag{A-68}$$

$$\begin{aligned} a_3 = \xi[&2(1+\alpha)(\beta+\gamma)\theta^3 + \{(2\gamma-3)\gamma + (1+\alpha)(3-2\gamma\theta - 2\beta\theta)\theta\}\theta^2 \\ &- 2\{2(1-\gamma)(1+\alpha)+(1-\gamma\theta)(\tfrac{1}{2}-\gamma)\}\theta+2\{(1-\gamma)(\tfrac{1}{2}-\gamma)-(\gamma-\beta-\tfrac{1}{2})\}] \end{aligned} \tag{A-69}$$

$$\begin{aligned} a_4 = {} & \beta\theta^4(1+\alpha)^2 + \beta\theta^3(1+\alpha)(\gamma-\frac{3}{2}) - (1+\alpha)^2(\beta\theta-\tfrac{1}{2})\theta^3 \\ & + (1+\alpha)[(\tfrac{1}{2}-\gamma)(\beta\theta-\tfrac{1}{2})-(1-\gamma)(1+\alpha)]\theta^2 \\ & + (1+\alpha)[(\tfrac{1}{2}-\gamma)(1-\gamma)-(\gamma-\beta-\tfrac{1}{2})]\theta + [\tfrac{1}{2}-\gamma][\gamma-\beta-\tfrac{1}{2}] \end{aligned} \tag{A-70}$$

For such a condition [i.e., Equation (A-65)] to hold

$$\xi = 0 \tag{A-71}$$

$$(1+\alpha)\theta + \gamma - \theta - \tfrac{1}{2} + 4\xi^2(\theta-1+\gamma) = 0 \tag{A-72}$$

or

$$\gamma = \tfrac{1}{2} - \alpha\theta \tag{A-73}$$

and

$$\beta = \frac{[(1+\alpha)(1-2\alpha)\theta + (\alpha-1)]\theta}{2[(1+\alpha)\theta^2-1]} \tag{A-74}$$

We observe at this point that the coefficients a_1, a_2, a_3, a_4 of Equation (A-66), given by Equations (A-67) through (A-70) assume the limiting values as given in References 15, 16, and 17.

Coef	Case $\theta = 1$, $\alpha+\gamma = \frac{1}{2}$	Case $\alpha = 0$, $\gamma = \frac{1}{2}$
a_1	2ξ	2ξ
a_2	$4\xi\gamma$	$2\xi^2(2\theta-1)$
a_3	$2\xi([illegible]-\alpha^2)$	$2\xi\left(\beta+\theta(\theta-1)\right)$
a_4	$-[illegible]([illegible]+[illegible])$	$\theta(\theta-1)\left[\frac{\theta}{2}-\beta(\theta+1)\right]$

(2) $A_3 = 0$ (Quadratic case)

In this case, the Routh-Hurwitz criterion of stability is

$$E_1 = 1 - 2A_1 + A_2 > 0 \tag{A-75}$$

$$E_2 = 1 - A_2 > 0 \tag{A-76}$$

$$E_3 = 1 + 2A_1 + A_2 > 0 \tag{A-77}$$

together with

$$A_3 = 0 \tag{A-78}$$

Equation (A-78), using (A-35) is equivalent to

$$[illegible] = 1 \tag{A-79}$$

$$[illegible][[illegible] - [illegible] + \tfrac{1}{2}] = 0 \tag{A-80}$$

and

$$E_1 = \frac{[illegible]2}{D} \tag{A-81}$$

$$E_2 = \frac{[illegible]}{D}[([illegible]+[illegible]-\tfrac{1}{2})[illegible] + 2[illegible]] \tag{A-82}$$

$$E_3 = \frac{1}{D}\left[\left\{2(1+\alpha)(2\beta-1) + 2(1-\gamma)\right\}\Omega^2 + 8(\gamma - \tfrac{1}{2})\xi\Omega + 4\right] \tag{A-83}$$

Equation (A-81) is always satisfied for real Ω, while Equation (A-82) implies

$$\gamma + \alpha - \tfrac{1}{2} \geq 0 \tag{A-84}$$

$$\gamma \geq 0 \tag{A-85}$$

Equation (A-83) results in

$$\gamma \geq \tfrac{1}{2} \tag{A-86}$$

and

$$(1+\alpha)(2\beta-1) + (1-\gamma) \geq 0 \tag{A-87}$$

Consider the cases

(a) $\alpha = 0$

Then Equation (A-87) is equivalent to

$$2\beta \geq \gamma \tag{A-88}$$

(b) $\alpha \neq 0$

By Equation (A-88), however, $\beta - \gamma + \tfrac{1}{2} = 0$

and Equation (A-87) becomes

$$2(1+2\alpha)\beta \geq (1+2\alpha) \tag{A-89}$$

If $1 + 2\alpha > 0$, then $\beta \geq \tfrac{1}{2}$, $\gamma \geq 1$, and $\gamma \geq \tfrac{1}{2} - \alpha$ (A-90)

If $1 + 2\alpha = 0$, then $0 \geq 1 + 2\alpha$ (A-91)

and it is satisfied as an equality

$$\alpha = -\tfrac{1}{2},\ \beta \geq \tfrac{1}{2},\ \text{and}\ \gamma \geq 1 \tag{A-92}$$

If $1 + 2\alpha < 0$, then $\beta \leq \frac{1}{2}$ (A-93)

But $\beta = \gamma - \frac{1}{2}$

Hence

$$\gamma - \tfrac{1}{2} \leq \tfrac{1}{2}$$

$$\tfrac{1}{2} - \alpha \leq \gamma \leq 1 \qquad \text{(A-94)}$$

which contradicts each other since $\alpha < -\frac{1}{2}$ (no solution).

If we further require the roots to be complex, then we must further satisfy

$$A_1^2 < A_2 \qquad \text{(A-95)}$$

where

$$2A_1 = \frac{1}{D}\left[\left\{(1+\alpha)(3\beta-\tfrac{1}{2}-\gamma)-\beta\right\}\Omega^2 + 2(2\gamma-1)\xi\Omega + 2\right]$$

$$= \frac{1}{D}\left[\left\{(1+\alpha)(2\beta-1) + (\tfrac{1}{2}-\gamma)\right\}\Omega^2 + 2(2\gamma-1)\xi\Omega + 2\right] \qquad \text{(A-96)}$$

$$A_2 = \frac{1}{D}\left[\left\{(1+\alpha)(3\beta-2\gamma) - (2\beta-\gamma-\tfrac{1}{2})\right\}\Omega^2 + 2(\gamma-1)\xi\Omega + 1\right]$$

$$= \frac{1}{D}\left[\left\{(1+\alpha)(\beta-1) + (\tfrac{3}{2}-\gamma)\right\}\Omega^2 + 2(\gamma-1)\xi\Omega + 1\right] \qquad \text{(A-97)}$$

where

$$D = 1 + 2\gamma\xi\Omega + (1+\alpha)\beta\Omega^2 \qquad \text{(A-98)}$$

Substituting Equations (A-96) and (A-97) in Equation (A-95) leads to

$$\left\{\left[-(1+\alpha)^2 + 2(1+\alpha)(2\beta-\gamma+\tfrac{1}{2}) - (\tfrac{1}{2}-\gamma)^2\right]\Omega^2 + [\gamma-\alpha-\tfrac{1}{2}]\xi\Omega + 4(1-\xi^2)\right\}\Omega^2 \geq 0 \qquad \text{(A-99)}$$

i.e.

$$|\xi| \leq 1 \qquad \text{(A-100)}$$

$$\gamma - \alpha - \tfrac{1}{2} \leq 0 \qquad \text{(A-101)}$$

$$-(1+\alpha)^2 + 2(1+\alpha)(2\beta-\gamma+\tfrac{1}{2}) - (\tfrac{1}{2}-\gamma)^2 \geq 0 \qquad \text{(A-102)}$$

or

$$\beta(\alpha+2) \geq \frac{\alpha}{2} + \tfrac{1}{2}(\tfrac{1}{2}+\gamma)^2 \tag{A-103}$$

Replacing $\beta = \gamma - \frac{1}{2}$ in Equation (A-102) we obtain

$$-\frac{1}{2}\gamma^2 + (\alpha + \frac{3}{2})\gamma - \frac{1}{8} - \frac{1}{2}(1+\alpha)(\alpha+2) \geq 0 \tag{A-104}$$

with roots

$$\gamma_{1,2} = \alpha + \frac{3}{2} \mp J\sqrt{-\alpha(\alpha+1)} \tag{A-105}$$

where

$$J = \sqrt{-1}$$

In the interval $-1 \leq \alpha \leq 0$, $\alpha = -1$, $\gamma = \frac{1}{2}$

$$\alpha = -\tfrac{1}{2}, \ \gamma = 1$$

$$\alpha = 0 \ , \ \gamma = \frac{3}{2}$$

satisfy (A-104).

We must check whether all the other requirements are valid. For the case $\alpha = -1$, $\gamma = \frac{1}{2}$ and $\beta = 0$, Equation (A-82) or (A-84) are not satisfied. Hence, this is not a solution. For $\alpha = -\frac{1}{2}$, $\gamma = 1$, $\beta = \frac{1}{2}$ (and $\theta = 1$), this is a solution.

For $\alpha = 0$, $\gamma = \frac{3}{2}$, $\beta = 1$ (and $\theta = 1$), this is a solution.

The fact that $\alpha = -1$, $\gamma = \frac{1}{2}$, $\beta = 0$ is not a solution can also be seen directly by inspecting Equations (A-109) through (A-112), where

$$2A_1 = \frac{-\beta\Omega^2 + 4(\gamma-\frac{1}{2})\xi\Omega + 2}{(1+2\gamma\xi\Omega)} \tag{A-106}$$

$$A_2 = \frac{-(2\beta-\gamma-\frac{1}{2})\Omega^2 + 2(\gamma-1)\xi\Omega + 1}{(1+2\gamma\xi\Omega)} \tag{A-107}$$

$$A_3 = \frac{(\gamma-\beta-\frac{1}{2})\Omega^2}{(1+2\gamma\xi\Omega)} \tag{A-108}$$

$$E_1 = 1 - 2A_1 + A_2 = \frac{(-\beta+\gamma+\frac{1}{2})\Omega^2}{1+2\gamma\xi\Omega} = \frac{\Omega^2}{(1+2\gamma\xi\Omega)} \geq 0 \tag{A-109}$$

since $\beta = \gamma - \frac{1}{2}$

$$E_2 = 1 - A_2 = \frac{(2\beta-\gamma-\frac{1}{2})\Omega^2 + 2\xi\Omega}{(1+2\gamma\xi\Omega)} = \frac{(\gamma-\frac{3}{2})\Omega^2 + 2\xi\Omega}{(1+2\gamma\xi\Omega)} \tag{A-110}$$

$$E_3 = 1 + 2A_1 + A_2 = \frac{-(3\beta-\gamma-\frac{1}{2})\Omega^2 + 8(\gamma-\frac{1}{2})\xi\Omega + 4}{(1+2\gamma\xi\Omega)} =$$

$$= \frac{-2(\gamma-1)\Omega^2 + 8(\gamma-\frac{1}{2})\xi\Omega + 4}{1+2\gamma\xi\Omega} \tag{A-111}$$

These conditions lead to

$$\left.\begin{array}{l} \gamma - \frac{3}{2} \geq 0 \\ -(\gamma-1) \geq 0 \\ \gamma - \frac{1}{2} \geq 0 \end{array}\right\} \quad \text{the first two are contradictory.} \tag{A-112}$$

Hence for $\theta = 1$, $\alpha = -1$ and $\beta = \gamma - \frac{1}{2}$ there is no unconditional stable range.

SUMMARY (CASE $A_3 = 0$)

CONDITIONS	
$\theta = 1$	$\alpha[\beta-\gamma+\frac{1}{2}] = 0$
$\gamma + \alpha - \frac{1}{2} \geq 0$	
$\gamma - \frac{1}{2} \geq 0$, $(1+\alpha)(2\beta-1) + (1-\gamma) \geq 0$	
$\xi \geq 0$	

SUBCASES

SUBCASE 1	SUBCASE 2		
$\alpha = 0$	$\alpha \neq 0$		
$2\beta \geq \gamma$ $\gamma \geq \frac{1}{2}$	$\beta - \gamma + \frac{1}{2} = 0$ $2(1+2\alpha)\beta \geq (1+2\alpha)$		
	SUBSUBCASE		
	$1 + 2\alpha > 0$ $\beta \geq \frac{1}{2}$ $\gamma \geq \frac{1}{2} - \alpha$	$1+2\alpha = 0$ $\gamma \geq 1$ $\beta \geq \frac{1}{2}$	$1+2\alpha < 0$ $\beta \leq \frac{1}{2}$ $\gamma \leq 1$ $-\alpha \leq \gamma \leq 1$ No Solution

ADDITIONAL REQUIREMENTS FOR ROOTS TO BE COMPLEX

$$|\xi| \leq 1$$

$$\gamma - \alpha - \tfrac{1}{2} \geq 0$$

$$(\alpha+2)\beta \geq \frac{\alpha}{2} + \tfrac{1}{2}(\tfrac{1}{2}+\gamma)^2$$

FOR SUBCASE 1	FOR SUBCASE 2
$\alpha = 0$	$\alpha = -1$
$\beta \geq \frac{1}{4}(\frac{1}{2}+\gamma)^2$	No solution
$\alpha = 0$ $\gamma = \frac{3}{2}$ $\beta = 1$ $\xi \geq 0$ a solution	

Observations

(1) For $\alpha = 0$, $\theta = 1$ we get the Newmark-β method, with the known conditions for unconditional stability (See Table 5-1.)

$$\gamma \geq \tfrac{1}{2} \tag{A-113}$$

$$\beta \geq \frac{\gamma}{2} \tag{A-114}$$

For complex roots we also have

$$\beta \geq \frac{1}{4}(\tfrac{1}{2}+\gamma)^2 \tag{A-115}$$

In the present case ($\alpha = 0$, $\theta = 1$) we observe the two subcases

(a) $\gamma = \tfrac{1}{2}$

with

$$A_1 = 1 - \frac{(2\xi\Omega+\Omega^2)}{2(1+\xi\Omega+\beta\Omega^2)} \tag{A-116}$$

$$A_2 = \frac{1-\xi\Omega+\beta\Omega^2}{[1+\xi\Omega+\beta\Omega^2]} = 1 - \frac{2\xi\Omega}{[1+\xi\Omega+\beta\Omega^2]} \tag{A-117}$$

$$A_1^2 - A_2 = \frac{-\Omega^2\left\{(4\beta-1)\Omega^2+4(1-\xi^2)\right\}}{4(1+\xi\Omega+\beta\Omega^2)^2} \tag{A-118}$$

and characteristic roots λ_1, λ_2

$$\begin{aligned}\lambda_1, \lambda_2 &= A_1 \pm \sqrt{A_1^2-A_2} \\ &= \left\{1 - \frac{(2\xi\Omega+\Omega^2)}{2(1+\xi\Omega+\beta\Omega^2)}\right\} \pm J\frac{\Omega\sqrt{(4\beta-1)\Omega^2+4(1-\xi^2)}}{2(1+\xi\Omega+\beta\Omega^2)}\end{aligned} \tag{A-119}$$

Observe from Equation (A-117) that when $\xi = 0$, $A_2 = 1$, i.e., the product of the roots (which for complex roots it is the square of the modulus) are unity, i.e., no artificial damping.

(b) $\gamma = \frac{3}{2}$ $\quad(\theta = 1,\ \alpha = 0)$

In this instance the product of the roots is

$$A_2 = 1 - \frac{[\Omega^2+2\xi\Omega]}{[1+3\xi\Omega+\beta\Omega^2]} \tag{A-120}$$

$$\text{As} \quad \Omega \rightarrow 0 \qquad A_2 \rightarrow 1 \tag{A-121}$$

$$\text{As} \quad \Omega \rightarrow \infty \qquad A_2 \rightarrow 1 - \frac{1}{\beta} \tag{A-122}$$

When $\xi = 0$, the magnitude of the complex roots becomes

$$|\lambda| = 1 - \frac{\Omega^2}{1+\beta\Omega^2} \tag{A-123}$$

Hence $$\lim_{\Omega \rightarrow \infty} |\lambda| = 1 - \frac{1}{\beta} \tag{A-124}$$

with dissipation at higher modes.

(2) For $\alpha = -\frac{1}{2}$, $\theta = 1$, $\gamma = 1$, $\beta = \frac{1}{2}$ $(\beta - \gamma + \frac{1}{2} = 0)$

$$2A_1 = -2\left\{\frac{\Omega^2-4\xi\Omega+4}{\Omega^2+8\xi\Omega+4}\right\} \tag{A-125}$$

$$A_2 = \frac{\Omega^2+4}{\Omega^2+8\xi\Omega+4} \tag{A-126}$$

$$A_3 = 0 \tag{A-127}$$

and

$$\lambda_{1,2} = \frac{1}{(\Omega^2+4)} \left\{ (\Omega^2-4) \pm 4j\Omega \right\} \tag{A-128}$$

Hence, we observe that for $\xi = 0$ $A_2 = 1$ (hence spectral radius is 1) and for this choice of parameters there is no artificial damping.

In the following, we give the form of the amplification matrix for very small and very large frequencies. We consider the following limiting cases.

1. When $p = \frac{\Delta t}{T} = 0$, $B = 0$, $\kappa = 0$

2. When $p = \frac{\Delta t}{T} = \infty$, $B = \frac{1}{(1+\alpha)\beta\theta^3}$, $\kappa = 0$

1. $p = 0$, $B = 0$, $\kappa = 0$

$$\frac{B}{\omega^2 \Delta t^2} \to \frac{1}{\theta}$$

and the amplification matrix assumes the form

$$\underset{\sim}{A} = \begin{bmatrix} 1 - \frac{1}{\theta} & 0 & 0 \\ 1 - \frac{\gamma}{\theta} & 1 & 0 \\ \frac{1}{2} - \frac{\beta}{\theta} & 1 & 1 \end{bmatrix} \tag{A-129}$$

2. $p = \infty$, $B = \frac{1}{(1+\alpha)\beta\theta^3}$, $\kappa = 0$

$$\frac{B}{\omega^2 \Delta t^2} \to 0$$

and

$$\underset{\sim}{A} = \begin{vmatrix} 1 - \frac{1}{2\beta\theta} & -\frac{1}{\beta\theta^2} & -\frac{1}{(1+\alpha)\beta\theta^3} \\ 1 - \frac{\gamma}{2\beta\theta} & 1 - \frac{\gamma}{\beta\theta^2} & -\frac{\gamma}{(1+\alpha)\beta\theta^3} \\ \frac{1}{2} - \frac{1}{2\theta} & 1 - \frac{1}{\theta^2} & 1 - \frac{1}{(1+\alpha)\theta^3} \end{vmatrix}$$

where $\omega = \frac{2\pi}{T}$

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